Correlations for the orthogonal-unitary and symplectic-unitary transitions at the hard and soft edges

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Abstract

For the orthogonal-unitary and symplectic-unitary transitions in random matrix theory, the general parameter dependent distribution between two sets of eigenvalues with two different parameter values can be expressed as a quaternion determinant. For the parameter dependent Gaussian and Laguerre ensembles the matrix elements of the determinant are expressed in terms of corresponding skew-orthogonal polynomials, and their limiting value for infinite matrix dimension are computed in the vicinity of the soft and hard edges respectively. A connection formula relating the distributions at the hard and soft edge is obtained, and a universal asymptotic behaviour of the two point correlation is identified.

1 Introduction

Random matrix ensembles for quantum Hamiltonians are defined in terms of the constraint imposed on the general Hermitian matrix representing the Hamiltonian by time reversal symmetry and spin-rotation invariance (when relevant). This leads to ten distinct random matrix ensembles [1]. The corresponding eigenvalue probability density function (p.d.f.) can be computed exactly when the independent elements are chosen to have a Gaussian distribution. One then finds that the ten distinct random matrix ensembles belong to one of six different eigenvalue p.d.f.'s. These are

$$\frac{1}{C_N}\prod_{l=1}^{N} e^{-\beta x_l^2/2}\prod_{1\le j<k\le N}|x_k-x_j|^{\beta} \tag{1.1}$$

which defines the Gaussian random matrix ensemble, or

$$\frac{1}{C_N'}\prod_{l=1}^N e^{-\beta x_l^2/2} x_l^{\beta a+1} \prod_{1\le j<k\le N} |x_k^2 - x_j^2|^\beta, \quad x_l > 0 \tag{1.2}$$

which defines the Laguerre random matrix ensemble, where $\beta = 1, 2$ or 4.

The study of these random matrix ensembles can be generalized to include a parameter τ. This allows the situation to be studied in which the system evolves between two distinct eigenvalue p.d.f.'s as τ is varied. For the Gaussian ensemble this can be achieved by choosing the joint distribution of the elements of the random matrix to be

$$P(X^{(0)}; X; \tau) = A_{\beta,\tau} \exp\Big(-\beta \mathrm{Tr}\Big\{ (X - \mathrm{e}^{-\tau} X^{(0)})^2 \Big\}/2|1 - \mathrm{e}^{-2\tau}| \Big). \tag{1.3}$$

In this expression X is a real symmetric ($\beta = 1$), Hermitian ($\beta = 2$) or self dual quaternion ($\beta = 4$) $N \times N$ matrix, and $X^{(0)}$ is a prescribed random matrix which must belong to a subspace of the symmetry class of X. For $\tau \to \infty$ the p.d.f. is independent of $X^{(0)}$ and is that of the standard Gaussian ensembles, with p.d.f. (1.1). For general τ Dyson [2] proved that the eigenvalue p.d.f. $p = p(x_1, \dots, x_N; \tau)$ satisfies the Fokker-Planck equation

$$\frac{\partial p}{\partial \tau} = \mathcal{L} p, \quad \mathcal{L} = \frac{1}{\beta}\sum_{j=1}^N \frac{\partial}{\partial x_j} e^{-\beta W} \frac{\partial}{\partial x_j} e^{\beta W} \tag{1.4}$$

with

$$W = W^{(H)} = \frac{1}{2}\sum_{j=1}^N x_j^2 - \sum_{1\le j<k\le N} \log|x_k - x_j| \tag{1.5}$$

(the superscript (H) is used because of a connection with the Hermite polynomials), subject to the initial condition that p agrees with the eigenvalue p.d.f. of $X^{(0)}$ at $\tau = 0$.

For the Laguerre ensemble, a parameter dependent theory can be developed (see e.g. [3]) by considering the non-negative matrices $A = X^\dagger X$, where X is an $n \times m$ parameter-dependent matrix

$$P(X^{(0)}; X; \tau) = A_{\beta,\tau} \exp\Big(-\beta \mathrm{Tr}\Big\{ (X - e^{-\tau} X^{(0)})^\dagger (X - e^{-\tau} X^{(0)}) \Big\}/2|1 - \mathrm{e}^{-2\tau}| \Big) \tag{1.6}$$

(c.f. (1.3)) and the elements of X are real ($\beta = 1$), complex ($\beta = 2$) or quaternion real ($\beta = 4$), and $X^{(0)}$ is a prescribed random matrix which must belong to a subspace of the symmetry class of X. For $\tau \to \infty$ the corresponding p.d.f. for the square of the eigenvalues of A is given by (1.2) with $a = n - m + 1 - 2/\beta$, while for general τ the eigenvalue p.d.f. $p = p(x_1^2, \dots, x_N^2; \tau)$ satisfies (1.4) with

$$W = W^{(L)} = \frac{1}{2}\sum_{j=1}^N x_j^2 - (a + 1/\beta)\sum_{j=1}^N \log x_j - \sum_{1\le j<k\le N} \log|x_k^2 - x_j^2| \tag{1.7}$$

(the superscript (L) is used because of a connection with the Laguerre polynomials).

Associated with the parameter dependent eigenvalue p.d.f.'s are parameter dependent distribution functions

$$\rho_{(n+m)}(x_1^{(1)}, \dots, x_n^{(1)}; \tau_1; x_1^{(2)}, \dots, x_m^{(2)}; \tau_2) \tag{1.8}$$

The distribution function $\rho_{(n+m)}$ is defined such that the ratio

$$\rho_{(n+m)}(x_1^{(1)}, \dots, x_n^{(1)}; \tau_1; x_1^{(2)}, \dots, x_m^{(2)}; \tau_2)/\rho_{(n+(m-1))}(x_1^{(1)}, \dots, x_n^{(1)}; \tau_1; x^{(2)}, \dots, x_{m-1}^{(2)}; \tau_2)$$

gives the density of eigenvalues at $x_m^{(2)}$ when the parameter equals τ_2, given that there were eigenvalues at $x_1^{(1)}, \dots, x_n^{(1)}$ when the parameter equalled τ_1, and that there are also eigenvalues at $x_1^{(2)}, \dots, x_m^{(2)}$ with parameter τ_2 (it is assumed $\tau_2 > \tau_1$).

For the parameter dependent Gaussian and Laguerre ensembles, quaternion determinant expressions have been derived for $\rho_{(n+m)}$ in the case that $\beta = 2$ in (1.4) and initial condition proportional to $e^{-\beta' W}$ with $\beta' = 1$ and 4 [7]. Because each β corresponds to a different symmetry class of the Hamiltonian (orthogonal, unitary and symplectic symmetry for $\beta = 1, 2$ and 4 respectively), this describes the transition from orthogonal or symplectic symmetry to unitary symmetry as the parameter τ is varied from 0 to ∞.

The determinant expressions apply for the finite system. As revised in Section 2, they involve skew orthogonal polynomials known from the study of the distribution functions for $\tau = 0$ in the $\beta = 1$ and 4 cases [5, 6], the orthogonal polynomials related to the $\tau = 0$, $\beta = 2$ case, as well as the elements of a connection matrix which specifies the orthogonal polynomials in terms of the appropriate skew orthogonal polynomials. In the Gaussian case, this matrix can be written down from knowledge of the expression for the skew orthogonal polynomials in terms of the orthogonal polynomials (the Hermite polynomials). In the Laguerre case this step requires some calculation which is undertaken below.

Once all these quantities are specified, the general $(n+m)$-point distribution function in the finite system is known explicitly. The immediate task is to compute the scaled thermodynamic limit. This is done in Section 3 for the Laguerre ensemble in the neighbourhood of the hard edge (the smallest eigenvalues, which are constrained to be positive), and in Section 4 for the Gaussian ensemble in the neighbourhood of the soft edge (the largest eigenvalues).

In Section 5 inter-relationships between the distribution functions are discussed. This is done by analyzing certain limits in (1.8): the limit $\tau_1, \tau_2 \to \infty$, $\tau_1 - \tau_2$ fixed; the $a \to \infty$ limit of the hard edge; and the bulk limit corresponding to large distances from the edge. Also considered is the asymptotic form of the dynamical density-density function $\rho^T_{(1+1)}(X, Y; t)$, which is shown to exhibit a universal non-oscillatory decay for $X, Y, t \to \infty$. A consequence of this decay is an O(1) fluctuation formula for the variance of a slowly varying linear statistic. In the final subsections a realization of the density at the soft edge for $\beta = 1$ initial conditions is given by an empirical computation based on the parameter dependent matrices (1.3) with $X^{(0)}$ a real symmetric matrix, and the explicit form of the static distribution functions at the hard and soft edges for $\beta = 1$ and 4 is noted.

In the Appendix it is shown how our results for the Laguerre ensemble with N finite coincide, in the static limit $\tau \to 0$, with formulas given recently by Widom [8].

2 Revision of the formalism

At the special coupling $\beta = 2$, the Fokker-Planck equation (1.4) with W given by (1.5) or (1.7) can be transformed into an imaginary-time Schrödinger equation for N independent fermions (see e.g. [3]). As such the exact N-body Green function $G(\{x_j^{(0)}\}, \{x_j\}; \tau)$ can be written as a determinant involving the corresponding single particle Green functions. In terms of the new variable $y_j = x_j$ in (1.1) and $y_j = x_j^2$ in (1.2) we have previously shown [4] that

$$G(x_1^{(0)}, \cdots, x_N^{(0)}; x_1, \cdots, x_N; \tau) dx_1 \cdots dx_N = G(y_1^{(0)}, \cdots, y_N^{(0)}; y_1, \cdots, y_N; \tau) dy_1 \cdots dy_N \quad (2.1)$$

where

$$\begin{aligned} & G(y_1^{(0)}, \cdots, y_N^{(0)}; y_1, \cdots, y_N; \tau) \\ = \ & \mathrm{e}^{E_0 \tau/2} \prod_{j=1}^{N} \sqrt{\frac{w(y_j)}{w(y_j^{(0)})}} \prod_{1 \le j < k \le N} \frac{(y_k - y_j)}{(y_k^{(0)} - y_j^{(0)})} \det[g(y_j^{(0)}, y_k; \tau)]_{j,k=1,\cdots,N} \end{aligned} \quad (2.2)$$

Here

$$g(y^{(0)}, y; \tau) = \sqrt{w(y^{(0)}) w(y)} \sum_{j=0}^{\infty} p_j(y^{(0)}) p_j(y) \mathrm{e}^{-\gamma_j \tau} \quad (2.3)$$

and $\{p_j(x)\}$ are the orthonormal polynomials associated with the weight function $w(x)$. In the Gaussian case

$$w(y) = \mathrm{e}^{-y^2}, p_n(y) = 2^{-n/2} \pi^{-1/4} (n!)^{-1/2} H_n(y), \quad \gamma_n = n + \frac{1}{2}, \quad (2.4)$$

where $H_n(y)$ denotes the Hermite polynomial, while for the Laguerre ensemble

$$w(y) = y^a \mathrm{e}^{-y}, \quad p_n(y) = (-1)^n \Big(\frac{\Gamma(n+1)}{\Gamma(a+n+1)} \Big)^{1/2} L_n^a(y), \quad \gamma_n = 2(n + \frac{a+1}{2}) \quad (2.5)$$

where $L_n^a(y)$ denotes the Laguerre polynomial. The factor $e^{E_0 \tau/2}$ cancels out the formula for the distribution function, so the explicit value of E_0 is not needed.

Knowledge of the Green function allows the p.d.f.

$$p(y_1^{(1)}, \cdots, y_N^{(2)}; \tau_1; y_1^{(2)}, \cdots, y_N^{(2)}; \tau_2)$$

for the event that there are eigenvalues at $y_1^{(1)}, \ldots, y_N^{(1)}$ when the parameter equals τ_1, and at $y_1^{(2)}, \ldots, y_N^{(2)}$ when the parameter is increased to τ_2 to be calculated. Explicitly

$$\begin{aligned} & p(y_1^{(1)}, \cdots, y_N^{(2)}; \tau_1; y_1^{(2)}, \cdots, y_N^{(2)}; \tau_2) \\ & \quad = \frac{1}{N!} \int_{I'} dy_1^{(0)} \cdots \int_{I'} dy_N^{(0)} \, p_0(y_1^{(0)}, \cdots, y_N^{(0)}) G(y_1^{(0)}, \cdots, y_N^{(0)}; y_1^{(1)}, \cdots, y_N^{(1)}; \tau_1) \\ & \quad \times G(y_1^{(1)}, \cdots, y_N^{(1)}; y_1^{(2)}, \cdots, y_N^{(2)}; \tau_2 - \tau_1). \end{aligned} \quad (2.6)$$

Here p_0 denotes the prescribed initial p.d.f. (in the y variables) and I' denotes the transformed domain (for convenience this will be omitted from subsequent formulas). Initial

p.d.f.'s with orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) symmetry are given by the functional forms

$$p_0(y_1, \cdots, y_N) \propto \begin{cases} \prod_{j=1}^{N} \sqrt{w(y_j)} \prod_{j<l}^{N} |y_j - y_l|, & \beta = 1 \\ \prod_{j=1}^{N/2} w(y_j) \delta(y_j - y_{j+N/2}) \prod_{j<l}^{N/2} |y_j - y_l|^4, & \beta = 4 \quad (N \text{ even}), \end{cases} \tag{2.7}$$

The parameter dependent distribution function $\rho_{(n+m)}$ is calculated from the p.d.f. (2.6) according to the formula

$$\rho_{(n+m)}(y_1^{(1)}, \cdots, y_N^{(1)}; \tau_1; y_1^{(2)}, \cdots, y_N^{(2)}; \tau_2) = \frac{N!}{(N-n)!} \frac{N!}{(N-m)!}$$
$$\times \int dy_{n+1}^{(1)} \cdots \int dy_N^{(1)} \int dy_{m+1}^{(2)} \cdots \int dy_N^{(2)} p(y_1^{(1)}, \cdots, y_N^{(2)}; \tau_1; y_1^{(2)}, \cdots, y_N^{(2)}; \tau_2) \tag{2.8}$$

In a recent work [7] we have shown that for initial conditions (2.7), and G specified by (2.2) with general weight function $w(y)$ and corresponding orthogonal polynomials $\{p_k(y)\}_{k=0,1,\dots}$, the quantity (2.8) can be expressed as a $(n+m)$-dimensional quaternion determinant Tdet (this quantity, which was introduced into random matrix theory by Dyson [9], is explicitly defined in e.g. [4]).

The entries of the quaternion determinant depend on the functions

$$S(x, y; \tau_x, \tau_y), \quad \tilde{I}(x, y; \tau_x, \tau_y), \quad D(x, y; \tau_x, \tau_y).$$

These functions in turn depend on the quantity $F(x, y; \tau)$, which is defined for the $\beta = 1$ initial conditions in (2.7) by

$$F(x, y; \tau) = \int dz' \int^{z'} dz \Big(g(x, z; \tau) g(y, z'; \tau) - g(y, z; \tau) g(x, z'; \tau) \Big), \tag{2.9}$$

while for the $\beta = 4$ initial conditions in (2.7) it is specified by

$$F(x, y; \tau) = \int dz \{ g(x, z; \tau) \frac{\partial}{\partial z} g(y, z; \tau) - g(y, z; \tau) \frac{\partial}{\partial z} g(x, z; \tau) \}. \tag{2.10}$$

They also depend on a family of skew-orthogonal monic polynomials $R_k(x; \tau)$ of degree k. The terminology skew-orthogonal means that they satisfy the skew orthogonality relations

$$< R_{2m}(\cdot; \tau), R_{2n+1}(\cdot; \tau) >= - < R_{2n+1}(\cdot; \tau), R_{2m}(\cdot; \tau) >= r_m(\tau) \delta_{m,n},$$
$$< R_{2m}(\cdot; \tau), R_{2n}(\cdot; \tau) >= 0, \quad < R_{2m+1}(\cdot; \tau), R_{2n+1}(\cdot; \tau) >= 0 \tag{2.11}$$

where

$$< f(x), g(y) >= \frac{1}{2} \int dx \int dy \sqrt{w(x) w(y)} F(y, x; \tau) \Big(f(y) g(x) - f(x) g(y) \Big). \tag{2.12}$$

The polynomials $R_k(x; \tau)$ can be deduced from their static counterparts [10, 4]. Suppose at $\tau = 0$ we write

$$R_n(x; 0) = \sum_{j=0}^{n} \alpha_{nj} C_j(x), \quad \alpha_{nn} = 1, \tag{2.13}$$

where the $C_j(x)$ are the monic version of the orthogonal polynomials $p_j(x)$ in (2.3). Then it is straightforward to verify that

$$R_n(x;\tau)e^{\gamma_n\tau} = \sum_{j=0}^{n} \alpha_{nj} C_j(x) e^{\gamma_j\tau} \tag{2.14}$$

and

$$r_k(\tau) = r_k(0)e^{-(\gamma_{2k}+\gamma_{2k+1})\tau} \tag{2.15}$$

Still another function involved is

$$\Phi_k(x;\tau) := \int F(y,x;\tau)\sqrt{w(y)}R_k(y;\tau)\,dy. \tag{2.16}$$

Now, if we invert (2.13) by writing

$$C_n(x)e^{\gamma_n\tau} = \sum_{j=0}^{n} \beta_{nj} e^{\gamma_j\tau} R_j(x;\tau), \quad \beta_{nn} = 1. \tag{2.17}$$

then $\Phi_k(x;\tau)$ can be expanded according to [4]

$$\begin{aligned} \Phi_{2k-1}(x;\tau)\mathrm{e}^{\gamma_{2k-1}\tau} &= -\sqrt{w(x)}r_{k-1}(0)\sum_{\nu=2k-2}^{\infty} \frac{C_\nu(x)e^{-\gamma_\nu\tau}}{h_\nu}\beta_{\nu\ 2k-2}, \\ \Phi_{2k-2}(x;\tau)e^{\gamma_{2k-2}\tau} &= \sqrt{w(x)}r_{k-1}(0)\sum_{\nu=2k-1}^{\infty} \frac{C_\nu(x)e^{-\gamma_\nu\tau}}{h_\nu}\beta_{\nu\ 2k-1}. \end{aligned} \tag{2.18}$$

where h_ν is the normalization

$$\int w(x)C_j(x)C_k(x)\,dx = h_j\delta_{j,k}. \tag{2.19}$$

With this notation, we can now define the functions $S, \tilde{I}$ and D. We have, assuming N even,

$$\begin{aligned} S(x,y;\tau_x,\tau_y) &:= \sqrt{w(y)} \\ &\times \sum_{l=0}^{N/2-1} \frac{e^{\gamma_{2l}\tau_x+\gamma_{2l+1}\tau_y}}{r_l(0)}\{\Phi_{2l}(x;\tau_x)R_{2l+1}(y;\tau_y) - \Phi_{2l+1}(x;\tau_x)R_{2l}(y;\tau_y)\}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\tilde{I}(x,y;\tau_x,\tau_y) \\ &:= \sum_{l=N/2}^{\infty} \frac{e^{\gamma_{2l}\tau_x+\gamma_{2l+1}\tau_y}}{r_l(0)}\{\Phi_{2l}(x;\tau)\Phi_{2l+1}(y;\tau) - \Phi_{2l+1}(x;\tau)\Phi_{2l}(y;\tau)\} \end{aligned} \tag{2.21}$$

$$\begin{aligned} D(x,y;\tau_x,\tau_y) &:= \sqrt{w(x)w(y)} \\ &\times \sum_{l=0}^{N/2-1} \frac{e^{\gamma_{2l}\tau_x+\gamma_{2l+1}\tau_y}}{r_l(0)}\{R_{2l}(x;\tau)R_{2l+1}(y;\tau) - R_{2l+1}(x;\tau)R_{2l}(y;\tau)\} \end{aligned} \tag{2.22}$$

and in terms of S we define

$$\tilde{S}(x,y;\tau_x,\tau_y) := S(x,y;\tau_x,\tau_y) - g(x,y;\tau_x-\tau_y), \quad \tau_x > \tau_y. \tag{2.23}$$

The formula for $\rho_{(n+m)}$ can now be stated. It reads [7]

$$\begin{aligned} &\rho_{(n+m)}(x_1^{(1)},\cdots,x_N^{(1)};\tau_1;x_1^{(2)},\cdots,x_N^{(2)};\tau_2) \\ &= \mathrm{Tdet}\begin{bmatrix} [f_1(x_j^{(1)},x_k^{(1)};\tau_1,\tau_1)]_{n\times n} & [f_2(x_j^{(1)},x_k^{(2)};\tau_1,\tau_2)]_{n\times m} \\ [f_2^D(x_j^{(1)},x_k^{(2)};\tau_1,\tau_2)]^T_{n\times m} & [f_1(x_j^{(2)},x_k^{(2)};\tau_2,\tau_2)]_{m\times m} \end{bmatrix} \end{aligned} \tag{2.24}$$

where

$$f_1(x,y;\tau,\tau) := \begin{bmatrix} S(x,y;\tau,\tau) & \tilde{I}(x,y;\tau,\tau) \\ D(x,y;\tau,\tau) & S(y,x;\tau,\tau) \end{bmatrix},$$

$$f_2(x,y;\tau_x,\tau_y) := \begin{bmatrix} S(x,y;\tau_x,\tau_y) & \tilde{I}(x,y;\tau_x,\tau_y) \\ D(x,y;\tau_x,\tau_y) & \tilde{S}(y,x;\tau_y,\tau_x) \end{bmatrix}$$

and

$$f_2^D(x,y;\tau_x,\tau_y) := \begin{bmatrix} \tilde{S}(y,x;\tau_y,\tau_x) & -\tilde{I}(x,y;\tau_x,\tau_y) \\ -D(x,y;\tau_x,\tau_y) & S(x,y;\tau_x,\tau_y) \end{bmatrix}.$$

If we denote the matrix in (2.24) by X, and note from the definitions (2.21) and (2.22) that

$$\tilde{I}(x,y;\tau_x,\tau_y) = -\tilde{I}(y,x;\tau_y,\tau_x), \quad D(x,y;\tau_x,\tau_y) = -D(y,x;\tau_y,\tau_x) \tag{2.25}$$

then we see that ZX where

$$Z = \mathbf{1}_{(n+m)} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is an even dimensional antisymmetric matrix. Its Pfaffian is thus well defined, and in fact [9]

$$\mathrm{Tdet}X = \mathrm{Pf}(ZX). \tag{2.26}$$

This formula can be used to compute the formula (2.24).

2.1 Alternative formulas and inter-relationships

As first noted in [4], substitution of (2.18) in (2.20) and use of (2.17) implies the decomposition

$$S(x,y;\tau_x,\tau_y) = S_1(x,y;\tau_x,\tau_y) + S_2(x,y;\tau_x,\tau_y) \tag{2.27}$$

where

$$S_1(x,y;\tau_x,\tau_y) = \sqrt{w(x)w(y)}\sum_{\nu=0}^{N-1}\frac{C_\nu(x)C_\nu(y)e^{-\gamma_\nu(\tau_x-\tau_y)}}{h_\nu} \tag{2.28}$$

$$S_2(x,y;\tau_x,\tau_y) = \sqrt{w(x)w(y)}\sum_{\nu=N}^{\infty}\sum_{k=0}^{N-1}\frac{C_\nu(x)e^{-\gamma_\nu\tau_x}}{h_\nu}\beta_{\nu k}R_k(y;\tau_y)\mathrm{e}^{\gamma_k\tau_y} \tag{2.29}$$

Since $0 \le \gamma_1 < \gamma_2 < \cdots$ (recall (2.4) and (2.5)) we see that for $N \to \infty$ and with all other parameters fixed, $S_2 \to 0$ while

$$S_1\left(x, y; \tau_x, \tau_y\right) \to g\left(x, y; \tau_x - \tau_y\right) \ ,$$

assuming $\tau_x - \tau_y > 0$. The use of this result is that it shows

$$\tilde{I}(x, y; \tau_x, \tau_y) = \int F(y', y; \tau_y)\Big(g(x, y'; \tau_x - \tau_y) - S(x, y'; \tau_x, \tau_y)\Big)\, dy' \qquad (2.30)$$

valid for $\tau_x - \tau_y > 0$ (for $\tau_x - \tau_y < 0$ we can first use the first equation in (2.25), then apply (2.30)).

We would also like to relate D to S. Apparently this can be done naturally only when $\tau_x = 0$. Using the fact that $g(x, y; 0) = \delta(x - y)$ we see from (2.9) and (2.10) that

$$F(x, y; 0) = \begin{cases} \mathrm{sgn}(y - x), & \beta = 1 \\ 2\frac{\partial}{\partial x}\delta(y - x), & \beta = 4. \end{cases} \qquad (2.31)$$

Recalling (2.16), these formulas show

$$\begin{aligned} \frac{d}{dx}\Phi_l(x; 0) &= 2\sqrt{w(x)}R_k(x; 0) \\ \Phi_l(x; 0) &= -2\frac{d}{dx}\Big(\sqrt{w(x)}R_k(x; 0)\Big) \end{aligned} \qquad (2.32)$$

in the two cases respectively. Thus for $\beta = 1$ initial conditions

$$\frac{d}{dx}S(x, y; 0, \tau_y) = -2D(x, y; 0, \tau_y) \qquad (2.33)$$

while for $\beta = 4$ initial conditions

$$\int_y^x S(x', y; 0, \tau_y)\, dx' = 2D(x, y; 0, \tau_y). \qquad (2.34)$$

3 The Laguerre ensemble

For the Laguerre ensemble the weight function $w(x)$ and constants γ_k are given by (2.5), while the monic orthogonal polynomials and corresponding normalization have the explicit form

$$C_n(x) = n!(-1)^n L_n^a(x), \quad h_n = \Gamma(n + 1)\Gamma(n + a + 1). \qquad (3.1)$$

This applies independent of the particular initial condition (orthogonal symmetry ($\beta = 1$) or symplectic symmetry ($\beta = 4$)). However, the skew orthogonal polynomials $R_k(x; \tau)$ and the quantities β_{jk} depend on the initial condition.

3.1 $\beta = 1$ initial conditions

In this case the monic skew orthogonal polynomials for $\tau = 0$ are known to be [5]

$$
\begin{aligned}
R_{2m}(x) &= -(2m)!\frac{d}{dx}L^a_{2m+1}(x) \;=\; (2m)!\sum_{j=0}^{2m}\frac{(-1)^j}{j!}C_j(x) \\
R_{2m+1}(x) &= -(2m+1)!L^a_{2m+1}(x) - (2m)!(a+2m+1)\frac{d}{dx}L^a_{2m}(x) \\
&= C_{2m+1}(x) + (2m)!(a+2m+1)\sum_{j=0}^{2m-1}\frac{(-1)^j}{j!}C_j(x). \qquad (3.2)
\end{aligned}
$$

with corresponding normalization

$$
r_n(0) = 4\Gamma(2n+1)\Gamma(a+2n+2). \qquad (3.3)
$$

These equations can readily be inverted. We find

$$
\begin{aligned}
C_{2m}(x) &= R_{2m}(x) + (2m+a)!\frac{m!}{(m+a/2)!}\sum_{j=0}^{m-1}\frac{(j+a/2+1)!(2j+2)!}{(j+1)!(2j+a+2)!} \\
&\quad \times\Big(\frac{1}{(2j+1)!}R_{2j+1}(x) - \frac{1}{(2j)!}R_{2j}(x)\Big) \\
C_{2m+1}(x) &= R_{2m+1}(x) + (2m+a+1)!\frac{m!}{(m+a/2)!}\sum_{j=0}^{m-1}\frac{(j+a/2+1)!(2j+2)!}{(j+1)!(2j+a+2)!} \\
&\quad \times\Big(\frac{1}{(2j+1)!}R_{2j+1}(x) - \frac{1}{(2j)!}R_{2j}(x)\Big) \qquad (3.4)
\end{aligned}
$$

(direct sustitution of (3.2) into the right hand sides verifies the validity of these formulas).

According to the definition (2.17) with $\tau = 0$, we read off from (3.4) that

$$
\begin{aligned}
\beta_{2m\,2m} &= \beta_{2m+1\,2m+1} = 1, \qquad \beta_{2m+1\,2m} = 0 \\
\beta_{\nu\,2j} &= -(\nu+a)!\frac{1}{(2j)!}\frac{[(\nu/2)]!}{([\nu/2]+a/2)!}\frac{(j+a/2+1)!}{(j+1)!}\frac{(2j+2)!}{(2j+a+2)!} \qquad (3.5) \\
\beta_{\nu\,2j+1} &= -\frac{1}{2j+1}\beta_{\nu\,2j}, \qquad (3.6)
\end{aligned}
$$

where $\nu > 2j+1$. A most significant feature of these formulas is that the dependence on p and l in β_{pl} separates. In particular, for $n > N-1$ and N even we have

$$
\beta_{pl} = \beta_{p\,N-1}\beta_{N-1\,l}, \quad 0 \le l \le N-1,\; l \ne N-2. \qquad (3.7)
$$

Substituting this in the formula (2.29) for S_2, and making use of the final formula in (3.5) in the case $(\nu, 2j+1) = (p, N-1)$ gives

$$
\begin{aligned}
S_2(x,y;\tau_x,\tau_y) &= \sqrt{w(x)w(y)}\sum_{p=N}^{\infty}\frac{C_p(x)}{h_p}e^{-\gamma_p\tau_x}\beta_{p\,N-1} \\
&\quad \times\Big(\sum_{l=0}^{N-1}\beta_{N-1\,l}R_l(y;\tau_y)e^{\gamma_l\tau_y} - (N-1)R_{N-2}(y;\tau_y)e^{\gamma_{N-2}\tau_y}\Big). \qquad (3.8)
\end{aligned}
$$

Now the sum over l can be computed using (2.17), while the sum over p is given by (2.18). Thus we have the explicit evaluation

$$\begin{aligned} S_2(x,y;\tau_x,\tau_y) &= \sqrt{w(x)}\Big(\frac{\Phi_{N-2}(x;\tau_x)e^{\gamma_{N-2}\tau_x}}{r_{N/2-1}(0)} - \sqrt{w(x)}\frac{C_{N-1}(x)e^{-\gamma_{N-1}\tau_x}}{h_{N-1}}\Big) \\ &\quad \times\Big(C_{N-1}(y)e^{\gamma_{N-1}\tau_y} - (N-1)R_{N-2}(y;\tau_y)e^{\gamma_{N-2}\tau_y}\Big). \end{aligned} \tag{3.9}$$

In Appendix A we will show that in the limit $\tau_x, \tau_y \to 0$ this expression reduces down to the form of S_2 recently derived by Widom [8].

3.1.1 The thermodynamic limit

In the neighbourhood of the spectrum edge at the origin, we will show that after introducing the scaled coordinates and parameters X and t according to

$$x = \frac{X}{4N} \qquad \tau = \frac{t}{2N} \tag{3.10}$$

(the numerical factors 1/4 and 1/2 are chosen for convenience), the distribution functions have a well defined large N limit. In particular, the scaled distribution function (2.8), to be denoted $\rho^{\rm hard}_{(n+m)}$, is given by

$$\begin{aligned} &\rho^{\rm hard}_{(n+m)}(X_1^{(1)},\dots,X_n^{(1)};t^{(1)};X_1^{(2)},\dots,X_m^{(2)};t^{(2)}) \\ &\quad = \lim_{N\to\infty}\Big(\frac{1}{4N}\Big)^{n+m}\rho_{(n+m)}\Big(\frac{X_1^{(1)}}{4N},\dots,\frac{X_n^{(1)}}{4N};\frac{t^{(1)}}{4N};\frac{X_1^{(2)}}{4N},\dots,\frac{X_m^{(2)}}{4N};\frac{t^{(2)}}{2N}\Big) \end{aligned} \tag{3.11}$$

From the formula (2.26) for Tdet it follows that $\rho^{\rm hard}_{(n+m)}$ is given by the formula (2.24) with the functions $S, \tilde{S}, \tilde{I}, D$ replaced by their scaled counterparts

$$S_1^{\rm hard}(X,Y;t_X,t_Y) := \lim_{N\to\infty}\frac{1}{4N}S\Big(\frac{X}{4N},\frac{Y}{4N};\frac{t_X}{2N},\frac{t_Y}{2N}\Big) \tag{3.12}$$

$$\tilde{S}_1^{\rm hard}(X,Y;t_X,t_Y) := \lim_{N\to\infty}\frac{1}{4N}\tilde{S}\Big(\frac{X}{4N},\frac{Y}{4N};\frac{t_X}{2N},\frac{t_Y}{2N}\Big) \tag{3.13}$$

$$\tilde{I}_1^{\rm hard}(X,Y;t_X,t_Y) := \lim_{N\to\infty}\tilde{I}\Big(\frac{X}{4N},\frac{Y}{4N};\frac{t_X}{2N},\frac{t_Y}{2N}\Big) \tag{3.14}$$

$$D_1^{\rm hard}(X,Y;t_X,t_Y) := \lim_{N\to\infty}\Big(\frac{1}{4N}\Big)^2 D\Big(\frac{X}{4N},\frac{Y}{4N};\frac{t_X}{2N},\frac{t_Y}{2N}\Big) \tag{3.15}$$

where the subscripts 1 indicate $\beta = 1$ initial conditions.

Note that it was permissable to multiply $\frac{1}{4N}\tilde{I}$ by $4N$ and divide $\frac{1}{4N}D$ by $4N$ in the above definitions because of an invariance property of the Tdet formula (2.24). Thus in general each 2×2 block Q_{jk} can be replaced by the 2×2 block

$$A^{-1}Q_{jk}A. \tag{3.16}$$

Choosing

$$A = \begin{pmatrix} 1/\sqrt{4N} & 0 \\ 0 & \sqrt{4N} \end{pmatrix} \tag{3.17}$$

then transforms $\frac{1}{4N}\tilde{I}$ and $\frac{1}{4N}D$ as required.

Asymptotics of $S(x, y; \tau_x, \tau_y)$

Let us first compute the asymptotic behaviour of $\Phi_{2k-2}(x;\tau)$. This is required in the computation of S_2 (with $2k = N$), and also $\tilde{I}$. From the expansion (2.18) and the explicit formulas (3.1) and (3.5) for the quantities $C_\nu(x)$, h_ν and $\beta_{\nu 2k-2}$ therein we have

$$\Phi_{2k-2}(x;\tau)e^{\gamma_{2k-2}\tau} = \sqrt{w(x)}r_{k-1}(0)\Big(- \frac{L^a_{2k-1}(x)}{(2k-1+a)!}e^{-\gamma_{2k-1}\tau} + \frac{(k+a/2)!(2k)!}{(2k-1)!(2k+a)!k!}\sum_{\nu=k}^{\infty}\frac{\nu!}{(\nu+a/2)!}\Big(L^a_{2\nu}(x)e^{-\gamma_{2\nu}\tau} - L^a_{2\nu+1}(x)e^{-\gamma_{2\nu+1}\tau}\Big)\Big). \quad (3.18)$$

We now proceed to estimate the summand for large ν.

From Stirling's formula

$$\frac{\nu!}{(\nu+a/2)!} \underset{\nu\to\infty}{\sim} \frac{1}{\nu^{a/2}}\,.$$

For the second factor in the summand we write

$$L^a_{2\nu}(x)e^{-\gamma_{2\nu}\tau} - L^a_{2\nu+1}(x)e^{-\gamma_{2\nu+1}\tau} = -e^{-\gamma_{2\nu+1}\tau}\left(L^a_{2\nu+1}(x) - L^a_{2\nu}(x) - L^a_{2\nu}(x)(e^{-\tau(\gamma_{2\nu}-\gamma_{2\nu+1})}-1)\right)\,. \quad (3.19)$$

Using the facts that

$$e^{\tau(\gamma_{2\nu}-\gamma_{2\nu+1})} = e^{-2\tau}\,, \qquad L^a_{2\nu+1}(x) - L^a_{2\nu}(x) = L^{a-1}_{2\nu+1}(x)\,, \quad (3.20)$$

and the asymptotic formula

$$\sqrt{w(x)}\,L^a_n(x) \underset{n\to\infty}{\sim} n^{a/2}J_a\left(2\sqrt{nx}\right) \quad (3.21)$$

where $J_a(z)$ denotes the Bessel function, we see that

$$\sqrt{w(x)}\,\frac{\nu!}{(\nu+a/2)!}\left(L^a_{2\nu}(x)e^{-\gamma_{2\nu}\tau} - L^a_{2\nu+1}(x)e^{-\gamma_{2\nu+1}\tau}\right)\Big|_{\substack{x=X/4N\\ \tau=t/2N}}$$
$$\sim \; -\nu^{-a/2}e^{-(2\nu/N)t}\Big(\sqrt{\frac{X}{4N}}(2\nu)^{(a-1)/2}J_{a-1}\Big(\sqrt{\frac{2jX}{N}}\Big) + \frac{t}{N}(2\nu)^{a/2}J_a\Big(\sqrt{\frac{2jX}{N}}\Big)\Big)$$
$$= \; -\frac{2^{a/2}}{N}u^{-a/2}\frac{d}{du}\left(e^{-ut}u^{a/2}J_a\left(\sqrt{uX}\right)\right), \qquad u := 2\nu/N\,. \quad (3.22)$$

(The equality is obtained by making use of a suitable differentiation formula for the Bessel function).

Substituting (3.22) in (3.18) gives the leading order behaviour of the sum over ν as a Riemann integral. Furthermore, use of (3.21) gives

$$\sqrt{w(x)}\,\frac{L^a_{2k-1}(x)}{(2k-1+a)!}\,e^{-\gamma_{2k-1}\tau}\Big|_{\substack{x=X/4N\\ \tau=t/2N}} \sim \frac{N^{-a/2}}{(2k-1)!}\,s^{-a/2}J_a\left(\sqrt{sX}\right)\,e^{-st}, \quad (3.23)$$

$s := 2k/N$, so in total we have

$$\Phi_{2k-2}\left(\frac{X}{4N};\frac{t}{2N}\right)e^{\gamma_{2k-2}t/2N} \sim r_{k-1}(0)\frac{N^{-a/2}}{(2k-1)!}\Big(\frac{1}{2}\left(1-2s^{-a/2}\right)J_a\left(\sqrt{sX}\right)e^{-st} -\frac{a}{4}\int_s^\infty \frac{e^{-ut}}{u}J_a\left(\sqrt{uX}\right)du\Big). \tag{3.24}$$

The expansions (3.23) and (3.24) with $2k = N$ and thus $s = 1$ suffice to specify the x-dependent factor in (3.9). Thus we have

$$\left(\frac{\Phi_{N-2}(x;\tau_x)}{r_{N/2-1}(0)} - \sqrt{w(x)}\,\frac{C_{N-1}(x)e^{-\gamma_{N-1}\tau_x}}{h_{N-1}}\right)\Bigg|_{\substack{x=X/4N\\ \tau_x=t_X/2N}} \sim \frac{N^{-a/2}}{2(N-1)!}\left(J_a\left(\sqrt{X}\right)e^{-t_X} - \frac{a}{2}\int_1^\infty \frac{e^{-ut_X}}{u}J_a\left(\sqrt{uX}\right)du\right). \tag{3.25}$$

Next we will compute the asymptotic form of $R_{2j}(y;\tau_y)$. In the case $2j = N-2$ this is required for the y-dependent factor in (3.9). From (2.14), (3.2) and (3.1) we see that

$$R_{2j}(y;\tau)\,e^{\gamma_{2j}\tau} = (2j)!\sum_{l=0}^{2j} L_l^a(y)\,e^{\gamma_l\tau}. \tag{3.26}$$

The asymptotic form of this expression is deduced by using (3.21) to approximate the summand, which leads to the result

$$\sqrt{w(y)}R_{2j}(y;\tau_y)e^{\gamma_{2j}\tau_y}\Bigg|_{\substack{y=Y/4N\\ \tau_y=t_Y/2N}} \sim (2j)!\,N^{1+a/2}\int_0^s v^{a/2}J_a\left(\sqrt{vY}\right)e^{vt_Y}\,dv, \tag{3.27}$$

where here $s = 2j/N$. Setting $s = 1$ and making further use of (3.23) we see that

$$\sqrt{w(y)}\,(C_{N-1}(y)e^{\gamma_{N-1}\tau_y} - (N-1)R_{N-2}(y;\tau_y)e^{\gamma_{N-2}\tau_y}) \sim -(N-1)!N^{1+a/2}\int_0^1 v^{a/2}J_a\left(\sqrt{vY}\right)e^{vt_Y}\,dv. \tag{3.28}$$

Combining (3.25) and (3.28) in the formula (3.9) for S_2 gives

$$S_2\left(\frac{X}{4N},\frac{Y}{4N};\frac{t_X}{2N},\frac{t_Y}{2N}\right) \sim \frac{N}{2}\left(-e^{-t_X}J_a\left(\sqrt{X}\right)+\frac{a}{2}\int_1^\infty \frac{e^{-ut_X}}{u}J_a\left(\sqrt{uX}\right)du\right) \times\left(\int_0^1 v^{a/2}J_a\left(\sqrt{vY}\right)e^{vt_Y}\,dv\right). \tag{3.29}$$

The asymptotics of S_1 can be determined by using (3.21) and Stirling's formula to approximate the integrand. This gives

$$S_1\left(\frac{X}{4N},\frac{Y}{4N};\frac{t_X}{2N},\frac{t_Y}{2N}\right) \sim N\int_0^1 J_a\left(\sqrt{uX}\right)\,J_a\left(\sqrt{uY}\right)\,e^{-u(t_X-t_Y)}\,du. \tag{3.30}$$

Since S is defined as the sum of S_1 and S_2, its asymptotic form is given by the sum of (3.29) and (3.30). Substituting into (3.12) gives

$$S_1^{\mathrm{hard}}(X,Y;t_X,t_Y) = \frac{1}{4}\int_0^1 J_a(\sqrt{uX})J_a(\sqrt{uY})e^{-u(t_X-t_Y)}\,du +\frac{1}{8}\Big(-e^{-t_X}J_a(\sqrt{X})+\frac{a}{2}\int_1^\infty \frac{e^{-ut_X}}{u}J_a(\sqrt{uX})\,du\Big)\int_0^1 dv\,J_a(\sqrt{vY})e^{t_Yv}v^{a/2}. \tag{3.31}$$

Asymptotics of $\tilde{I}(x, y; \tau_x, \tau_y)$

So as to induce a cancellation at leading order, we first perform some minor manipulation to the definition (2.21), by way of subtracting and adding $(2l+1)e^{\gamma_{2l}(\tau_x+\tau_y)}\Phi_{2l}(x;\tau_x)\Phi_{2l}(y;\tau_y)$ to the summand, so that it reads

$$\begin{aligned}\tilde{I}(x, y; \tau_x, \tau_y) \quad &= \sum_{l=N/2}^{\infty} \frac{1}{r_l(0)} \{e^{\gamma_{2l}\tau_x}\Phi_{2l}(x;\tau_x)\left(e^{\gamma_{2l+1}\tau_y}\Phi_{2l+1}(y;\tau_y)\right. \\ &\quad \left.-(2l+1)e^{\gamma_{2l}\tau_y}\Phi_{2l}(y;\tau_y)\right) - (x \leftrightarrow y)\} \,. \end{aligned} \tag{3.32}$$

The asymptotics of $e^{\gamma_{2l}\tau_x}\Phi_{2l}(x;\tau_x)$ can be read off from (3.24). For the y-dependent factor of the first term of the summand we note from the definitions (2.18) and the explicit formulas (3.1) and (3.5) that

$$\begin{aligned}&e^{\gamma_{2l+1}\tau}\Phi_{2l+1}(y;\tau) - (2l+1)e^{\gamma_{2l}\tau}\Phi_{2l}(y;\tau) \\ &\quad = \sqrt{w(y)}\, r_l(0) \left((2l+1)\frac{L^a_{2l+1}(y)e^{-\gamma_{2l+1}\tau}}{(2l+1+a)!} - \frac{L^a_{2l}(y)e^{-\gamma_{2l}\tau}}{(2l+a)!} \right) .\end{aligned}$$

Thus, using (3.3) and the asymptotic formula (3.21) we have

$$\begin{aligned}&\left(e^{\gamma_{2l+1}\tau}\Phi_{2l+1}(y;\tau) - (2l+1)e^{\gamma_{2l}\tau}\Phi_{2l}(y;\tau)\right)\Big|_{\substack{y=Y/4N \\ \tau=t/2N}} \\ &\quad \sim 4(2l+1)!N^{-1+a/2}\left\{\frac{d}{ds}\left(e^{-st}J_a\left(\sqrt{sY}\right)\right) - as^{-1+a/2}J_a\left(\sqrt{sY}\right)e^{-st}\right\}, \quad s = 2l/N \,.\end{aligned}$$

Substituting this result, together with (3.24) (after replacing k by $l+1$) in (3.32) shows that

$$\begin{aligned}\tilde{I}^{\rm hard}_1(X, Y; t_X, t_Y) = &\int_1^\infty ds \left\{ \left(1 - \frac{2}{s^{a/2}}\right) J_a\left(\sqrt{sX}\right) e^{-st_X} - \frac{a}{4}\int_s^\infty \frac{e^{-ut_X}}{u} J_a\left(\sqrt{uX}\right) du \right\} \\ &\times \left\{ \frac{d}{ds}\left(e^{-st_Y}J_a\left(\sqrt{sY}\right)\right) - as^{-1+a/2}J_a\left(\sqrt{sY}\right)e^{-st_Y} \right\} - (X \leftrightarrow Y)\end{aligned} \tag{3.33}$$

Asymptotics of $g(x, y; \tau)$

According to (2.3) and (2.5)

$$g(x, y; \tau) = (xy)^{a/2}e^{-(x+y)/2}\sum_{j=0}^{\infty}\frac{j!}{(j+a)!}L^a_j(x)L^a_j(y)e^{-2(j+(a+1)/2)\tau} \,. \tag{3.34}$$

This summation can be evaluated in closed form [11], giving

$$g(x, y; \tau) = e^{-(x+y)/2}e^{-(2a+1)\tau}\left(1 - e^{-2\tau}\right)^{-1}\exp\left(-\frac{e^{-2\tau}}{1-e^{-2\tau}}(x+y)\right) I_a\left(2\frac{\sqrt{xy}\,e^{-\tau}}{1-e^{-2\tau}}\right) ,$$

where I_a denotes the Bessel function of pure imaginary argument. Consequently

$$g\left(\frac{X}{4N}, \frac{Y}{4N}; \frac{t_X - t_Y}{2N}\right) \sim \frac{N}{t_X - t_Y}\exp\left(-\frac{X+Y}{t_X - t_Y}\right) I_a\left(\frac{\sqrt{XY}}{2(t_X - t_Y)}\right) . \tag{3.35}$$

Alternatively, proceeding as in the derivation of (3.30) we can deduce from (3.34) that

$$g\Big(\frac{X}{4N},\frac{Y}{4N};\frac{t_X-t_Y}{2N}\Big)\sim N\int_0^\infty J_a(\sqrt{uX})J_a(\sqrt{uY})e^{-u(t_X-t_Y)}\,du. \tag{3.36}$$

The immediate merit of this representation is that when substituted in (2.23), partial cancellation occurs with (3.30). Adding the result to (3.29) and substituting in (3.13) gives

$$\begin{aligned}\tilde S_1^{\rm hard}(X,Y;t_X,t_Y) &= -\frac{1}{4}\int_1^\infty J_a(\sqrt{uX})J_a(\sqrt{uY})e^{-u(t_X-t_Y)}\,du\\ &+\frac{1}{8}\Big(-e^{-t_X}J_a(\sqrt{X})+\frac{a}{2}\int_1^\infty\frac{e^{-ut_X}}{u}J_a(\sqrt{uX})\,du\Big)+\int_0^1 dv\,J_a(\sqrt{vY})e^{t_Y v}v^{a/2}\end{aligned}\tag{3.37}$$

Asymptotics of $D(x,y;\tau_x,\tau_y)$

According to (2.22) D is defined in terms of $\{R_j(x;\tau)\}$. Now $R_{2j}(y;\tau)$ is specified by (3.26). Deriving the analogous formula for $R_{2j+1}(x;\tau)$, which follows from (3.2), (2.14) and (3.1), we see that

$$\begin{aligned}R_{2j+1}(x;\tau)e^{\gamma_{2j+1}\tau} &= -(2j+1)!L^a_{2j+1}(x)e^{\gamma_{2j+1}\tau}-(2j)!(2j+a+1)L^a_{2j}(x)e^{\gamma_{2j}\tau}\\ &\quad+(2j+a+1)R_{2j}(x;\tau)e^{\gamma_{2j}\tau}\,.\end{aligned}$$

Hence

$$\begin{aligned}&(e^{\gamma_{2j+1}\tau_x}R_{2j+1}(x;\tau_x)e^{\gamma_{2j}\tau_y}R_{2j}(y;\tau_y)-(x\leftrightarrow y))=\\ &\quad\Big(-(2j+1)!L^a_{2j+1}(x)e^{\gamma_{2j+1}\tau_x}-(2j)!(2j+a+1)L^a_{2j}(x)e^{\gamma_{2j}\tau_x}\Big)R_{2j}(y;\tau_y)e^{\gamma_{2j}\tau_y}-(x\leftrightarrow y)\,.\end{aligned}$$

Now the asymptotic form of $R_{2j}(y;\tau_y)$ is specified by (3.27) while we see from (3.21) that

$$\begin{aligned}&\sqrt{w(x)}\Big(-(2j+1)!L^a_{2j+1}(x)e^{\gamma_{2j+1}\tau_x}-(2j)!(2j+a+1)L^a_{2j}(x)e^{\gamma_{2j}\tau_x}\Big)\Big|_{\substack{x=X/4N\\ \tau_x=t_X/2N}}\\ &\qquad\sim -2(2j+1)!e^{st_X}(Ns)^{a/2}J_a\left(\sqrt{sX}\right)\,,\qquad s=2j/N\,.\end{aligned}$$

Making use also of the explicit form (3.3) of $r_n(0)$ and using the definition (3.15) we thus find

$$\begin{aligned}D_1^{\rm hard}(X,Y;t_X,t_Y) &= -4^{-3}\\ &\times\Big\{\int_0^1 ds\,e^{st_X}s^{-a/2}J_a\left(\sqrt{sX}\right)\int_0^s dv\,e^{vt_Y}v^{a/2}J_a\left(\sqrt{vY}\right)-(X\leftrightarrow Y)\Big\}\,.\end{aligned}\tag{3.38}$$

3.2 $\beta=4$ initial conditions

The method of evaluation of the quantities S, $\tilde I$, and D for the Laguerre ensemble in the case of initial conditions with symplectic symmetry ($\beta=4$) closely parallels that just given for $\beta=1$ initial conditions. First, we note from previous work [5] that the monic

skew orthogonal polynomials at $\tau = 0$ have the explicit form

$$\begin{aligned}
R_{2j}(x) &= 2^{2j} j!\,(j+(a-1)/2)! \sum_{l=0}^{j} \frac{(2l)!}{l!\,2^{2l}} \frac{1}{(l+(a-1)/2)!} L_{2l}^{a-1}(x) \\
&= 2^{2j} j!\,(j+(a-1)/2)! \sum_{l=0}^{j} \frac{1}{l!\,2^{2l}\,(l+(a-1)/2)!} \left(C_{2l}(x) + 2l C_{2l-1}(x)\right) , \\
R_{2j+1}(x) &= -(2j+1)!\, L_{2j+1}^{a-1}(x) \\
&= C_{2j+1}(x) + (2j+1) C_{2j}(x) ,
\end{aligned} \tag{3.39}$$

with corresponding normalization

$$r_n(0) = \Gamma(2n+2)\,\Gamma(a+2n+1) . \tag{3.40}$$

The inversion of these equations gives that the quantities $\beta_{n\,j}$ in (2.17) have the explicit values

$$\begin{aligned}
\beta_{2m\,2m} &= \beta_{2m+1\,2m+1} = 1 , \qquad \beta_{2m+1\,2m} = -(2m+1) , \\
\beta_{\nu\,2j+1} &= (-1)^{\nu+1} \frac{\nu!}{(2j+1)!} , \qquad \beta_{\nu\,2j} = a\,\beta_{\nu\,2j+1} ,
\end{aligned} \tag{3.41}$$

for $\nu > 2j+1$. Thus the factorization formula (3.7) again holds, which when substituted in (2.29) and after making use of the explicit value of $\beta_{N-1\,N-2}$ and $\beta_{p\,N-1}$, and (2.17), implies

$$\begin{aligned}
S_2(x,y;\tau_x,\tau_y) &= \sqrt{w(x)w(y)} \sum_{p=N}^{\infty} \frac{C_p(x)}{h_p}\, e^{-\gamma_p \tau_x}\, \beta_{p\,N-1} \{ C_{N-1}(y) e^{\gamma_{N-1}\tau_y} \\
&\qquad + (a+N-1) R_{N-2}(y;\tau_y) e^{\gamma_{N-2}\tau_y} \} .
\end{aligned}$$

The sum over p is, after minor manipulation, formally identical to the sum over p in (3.8), so it can be summed using (2.18). Thus S_2 has the explicit form

$$\begin{aligned}
S_2(x,y;\tau_x,\tau_y) &= \sqrt{w(y)} \left(\frac{\Phi_{N-2}(x;\tau_x)}{r_{N/2-1}(0)} e^{\gamma_{N-2}\tau_x} - \sqrt{w(x)} \frac{C_{N-1}(x)}{h_{N-1}} e^{-\gamma_{N-1}\tau_x} \right) \\
&\quad \times \left(C_{N-1}(y) e^{\gamma_{N-1}\tau_y} + (a+N-1) R_{N-2}(y;\tau_y) e^{\gamma_{N-2}\tau_y} \right) .
\end{aligned} \tag{3.42}$$

As with (3.9), in the limit $\tau_x, \tau_y \to 0$ this reduces down to the form of S_2 derived by Widom [8] in the static $\beta = 4$ theory. The details of the verification are given in Appendix A.

3.2.1 The thermodynamic limit

Although the scaled distribution function is still given by (3.11), the required scaling of the elements of the quaternion determinant (2.24) differs slightly from the formulas (3.12)–(3.15) appropriate with $\beta = 1$ initial conditions. In particular (3.14) and (3.15) should now read

$$\tilde{I}_4^{\rm hard}(X,Y;t_X,t_Y) := \lim_{N\to\infty} \Big(\frac{1}{4N}\Big)^2 \tilde{I}\Big(\frac{X}{4N}, \frac{Y}{4N}; \frac{t_X}{2N}, \frac{t_Y}{2N}\Big) \tag{3.43}$$

$$D_4^{\rm hard}(X,Y;t_X,t_Y) := \lim_{N\to\infty} D\Big(\frac{X}{4N}, \frac{Y}{4N}; \frac{t_X}{2N}, \frac{t_Y}{2N}\Big) \tag{3.44}$$

while (3.12) and (3.13) remain the same, except that the subscripts 1 on the right hand sides should be replaced by 4.

Asymptotics of $S(x, y; \tau_x, \tau_y)$

From the expansion (2.18) and the explicit formulas (3.1) and (3.41) we have

$$\Phi_{2k-2}(x;\tau)e^{\gamma_{2k-2}\tau} = -\frac{r_{k-1}(0)\sqrt{w(x)}}{(2k-1)!}\sum_{\nu=2k-1}^{\infty}\frac{L_\nu^a(x)\,\nu!}{(\nu+a)!}\,e^{-\gamma_\nu\tau}\,.$$

Applying the asymptotic expansion (3.21) and Stirling's formula shows that

$$\Phi_{2k-2}\left(\frac{X}{4N};\frac{t}{2N}\right)e^{\gamma_{2k-2}t/2N} \sim -\frac{r_{k-1}(0)}{(2k-1)!}N^{1-a/2}\int_u^\infty dr\, r^{-a/2}e^{-rt}J_a\left(\sqrt{rX}\right)\,, \qquad (3.45)$$

where $u := 2k/N$. Setting $u = 1$ gives the asymptotic form of $\Phi_{N-2}(x;\tau_x)/r_{N/2-1}(0)$ as required in (3.42), and comparison with (3.23) shows that this term dominates the one involving $C_{N-1}(x)$. Thus here the formula (3.25) applies with the right hand side replaced by

$$-\frac{N^{1-a/2}}{(N-1)!}\int_u^\infty dr\, r^{-a/2}e^{-rt}J_a\left(\sqrt{rX}\right)\,. \qquad (3.46)$$

For the y-dependent factor in (3.42), we note from (2.14), (3.1), and (3.26) that

$$R_{2j}(y;\tau)e^{\gamma_{2j}\tau} = \\ 2^{2j}j!\,(j+(a-1)/2)!\sum_{l=0}^{j}\frac{(2l)!}{l!\,2^{2l}}\frac{1}{(l+(a-1)/2)!}\left(L_{2l}^a(y)e^{\gamma_{2l}\tau} - L_{2l-1}^a(y)e^{\gamma_{2l-1}\tau}\right).$$

Use of (3.21) and Stirling's formula gives for the corresponding asymptotic form

$$\frac{1}{(2j+1)!}\sqrt{w(y)}R_{2j}(y;\tau)e^{\gamma_{2j}\tau}\Big|_{\substack{y=Y/4N\\ \tau=t/2N}} \\ \sim \frac{N^{a/2-1}}{2}s^{-1+a/2}\int_0^s u^{-a/2}\frac{d}{du}\left(e^{ut}u^{a/2}J_a\left(\sqrt{uY}\right)\right)du\,, \quad s := 2j/N\,. \qquad (3.47)$$

Adding this with $s = 1$ to the asymptotic form of $\sqrt{w(y)}C_{N-1}(y)e^{\gamma_{N-1}\tau_y}/(N-1)!$ deduced from (3.23), and multiplying the result by (3.46) gives that

$$S_2\left(\frac{X}{4N},\frac{Y}{4N};\frac{t_X}{2N},\frac{t_Y}{2N}\right) \sim -N\int_1^\infty dr\, r^{-a/2}e^{-rt_X}J_a\left(\sqrt{rX}\right) \\ \times\left(-J_a\left(\sqrt{Y}\right)e^{t_Y} + \frac{1}{2}\int_0^1 u^{-a/2}\frac{d}{du}\left(e^{ut_Y}u^{a/2}J_a\left(\sqrt{uY}\right)\right)du\right)\,. \qquad (3.48)$$

The asymptotic form of S_1 can be read off from (3.30) since according to its definition (2.28), its value is the same for $\beta = 1$ and 4. The asymptotic form of S itself is thus given by the sum of (3.30) and (3.48). Thus we have

$$S_4^{\rm hard}(X,Y;t_X,t_Y) = \\ \frac{1}{4}\int_0^1 J_a(\sqrt{uX})J_a(\sqrt{uY})e^{-u(t_X-t_Y)}\,du - \frac{1}{4}\int_1^\infty dr\, r^{-a/2}e^{-rt_X}J_a(\sqrt{rX}) \\ \times\Big(-J_a(\sqrt{Y})e^{t_Y} + \frac{1}{2}\int_0^1 u^{-a/2}\frac{d}{du}\Big(e^{ut_Y}u^{a/2}J_a(\sqrt{uY})\Big)du\Big) \qquad (3.49)$$

Asymptotics of $\tilde{S}(x, y; \tau_x, \tau_y)$

The single particle Green function g as specified by (2.3) is independent of the initial condition, so me can use the formula (3.36). Dividing by $4N$ and subtracting from (3.49) shows

$$\begin{aligned}
&\tilde{S}_4^{\rm hard}(X, Y; t_X, t_Y) \\
&\quad = -\frac{1}{4}\int_1^\infty J_a(\sqrt{uX})J_a(\sqrt{uY})e^{-u(t_X - t_Y)}\,du - \frac{1}{4}\int_1^\infty dr\, r^{-a/2}e^{-rt_X}J_a(\sqrt{rX}) \\
&\quad \times\Big(- J_a(\sqrt{Y})e^{t_Y} + \frac{1}{2}\int_0^1 u^{-a/2}\frac{d}{du}\Big(e^{ut_Y}u^{a/2}J_a(\sqrt{uY})\Big)du\Big) \qquad (3.50)
\end{aligned}$$

Asymptotics of $\tilde{I}(x, y; \tau_x, \tau_y)$

The definition (2.21) gives that $\tilde{I}$ is defined in terms of $\{\Phi_l(x; \tau)\}$. The asymptotic form of $\Phi_{2k-2}(x; \tau)$ is given by (3.45). Instead of deriving an analogous expression for $\Phi_{2k-1}(y; \tau)$, we note from (2.18), (3.41), (3.1) and the formula (3.45) for Φ_{2k-2} that

$$\begin{aligned}
&\Phi_{2k-1}(y; \tau)e^{\gamma_{2k-1}\tau} = \\
&- \quad \frac{\sqrt{w(y)}\, r_{k-1}(0)}{(2k-2+a)!}\left(L^a_{2k-2}(y)e^{-\gamma_{2k-2}\tau} + L^a_{2k-1}(y)e^{-\gamma_{2k-1}\tau}\right) - a\Phi_{2k-2}(y; \tau)\,. \quad (3.51)
\end{aligned}$$

Thus

$$\begin{aligned}
&e^{\gamma_{2l}\tau_x}\Phi_{2l}(x; \tau_x)e^{\gamma_{2l+1}\tau_y}\Phi_{2l+1}(y; \tau_y) - (x \leftrightarrow y) = \\
&\quad e^{\gamma_{2l}\tau_x}\Phi_{2l}(x; \tau_x)\left(-\frac{\sqrt{w(y)}\, r_l(0)}{(2l+a)!}\right)\left(L^a_{2l}(y)e^{-\gamma_{2l}\tau_y} + L^a_{2l+1}(y)e^{-\gamma_{2l+1}\tau_y}\right) - (x \leftrightarrow y).
\end{aligned} \tag{3.52}$$

According to (3.21) and (3.40), the product of the second and third factors in the first term have the same leading asymptotic behaviour

$$-\,(2l+1)!\,4(vN)^{a/2}J_a\left(\sqrt{vY}\right)\,e^{-v\tau}\,, \quad v = 2l/N\,. \tag{3.53}$$

Multiplying this by (3.45) (with $k \mapsto l+1$) and substituting in (2.21) gives that the desired asymptotic form is

$$\begin{aligned}
&I_4^{\rm hard}(X, Y; t_X, t_Y) \\
&\quad = \frac{1}{4}\int_1^\infty \left\{\int_v^\infty dr\, r^{-a/2}e^{-rt_X}J_a\left(\sqrt{rX}\right)\right\}v^{a/2}J_a\left(\sqrt{vY}\right)e^{-vt_Y}\,dv - (X \leftrightarrow Y).
\end{aligned} \tag{3.54}$$

Asymptotics of $D(x, y; \tau_x, \tau_y)$

Here we require the asymptotic form of $\{R_l(x; \tau)\}$. For l even this is given by (3.47). On the other hand, for l odd we see from (2.14), (3.1) and (3.39) that

$$R_{2j+1}(y; \tau)e^{\gamma_{2j+1}\tau} = -(2j+1)!\left(L^a_{2j+1}(y)e^{\gamma_{2j+1}\tau} - L^a_{2j}(y)e^{\gamma_{2j}\tau}\right)\,. \tag{3.55}$$

Use of (3.21) shows

$$\sqrt{w(y)}\, R_{2j+1}(y;\tau)e^{\gamma_{2j+1}\tau}\Big|_{\substack{y=Y/4N\\ \tau=t/2N}} \sim -(2j+1)!N^{a/2-1}\frac{d}{ds}\left(e^{st}s^{a/2}J_a\left(\sqrt{sY}\right)\right), \tag{3.56}$$

$s := 2j/N$. Multiplying this by $(2j+1)!$ times (3.47) (with Y replaced by X) and dividing by (3.40) we thus have

$$\begin{aligned} D_4^{\rm hard}(X,Y;t_X,t_Y) \;\; &\sim \;\; -\frac{1}{4}\int_0^1 ds\, s^{-a/2}\left\{\int_0^s u^{-a/2}\frac{d}{du}\left(e^{ut_X}u^{a/2}J_a\left(\sqrt{uX}\right)\right)du\right\} \\ &\qquad \times\frac{d}{ds}\left(e^{st_Y}s^{a/2}J_a\left(\sqrt{sY}\right)\right) - (X\leftrightarrow Y)\,. \end{aligned} \tag{3.57}$$

4 Gaussian ensemble

For the Gaussian ensemble, we see from (2.4) that the corresponding monic orthogonal polynomials and normalization are

$$C_n(x) = 2^{-n}\, H_n(y), \qquad h_n = 2^{-n}\,\pi^{1/4}\, n!\,. \tag{4.1}$$

The skew orthogonal polynomials for both $\beta = 1$ and $\beta = 4$ initial conditions at $\tau = 0$ are known in terms of the $C_n(x)$ (see e.g. [12]). Using these expansions, the general formula (2.14) then gives

$$R_{2m}(x;\tau)e^{\gamma_{2m}\tau} = C_{2m}(x)e^{\gamma_{2m}\tau} \tag{4.2}$$
$$R_{2m+1}(x;\tau)e^{\gamma_{2m+1}\tau} = C_{2m+1}(x)e^{\gamma_{2m+1}\tau} - mC_{2m-1}(x)e^{\gamma_{2m-1}\tau} \tag{4.3}$$
$$r_m(0) = 2^{-2m+1}\sqrt{\pi}\Gamma(2m+1) \tag{4.4}$$

for $\beta = 1$ initial conditions, and

$$R_{2m}(x;\tau)e^{\gamma_{2m}\tau} = m!\sum_{l=0}^{m} e^{\gamma_{2l}\tau}\frac{C_{2l}(x)}{l!}$$
$$R_{2m+1}(x;\tau)e^{\gamma_{2m+1}\tau} = C_{2m+1}(x)e^{\gamma_{2m+1}\tau} \tag{4.5}$$
$$r_m(0) = 2^{-2m}\sqrt{\pi}\Gamma(2m+2) \tag{4.6}$$

for $\beta = 4$ initial conditions. Inverting these formulas for $\{R_j(x)\}$ in terms of $\{C_l(x)\}$ according to (2.17) gives

$$\beta_{2m\,j} = \delta_{2m,j}, \qquad \beta_{2m+1\,j} = \begin{cases} m!/l!, & j = 2l+1,\ l = 0,1,\dots,m \\ 0, & j = 2l,\ l = 0,1,\dots,m \end{cases} \tag{4.7}$$

for $\beta = 1$, while for $\beta = 4$ we find

$$\beta_{2m\,j} = \begin{cases} 1, & j = 2m \\ -m, & j = 2m-2 \\ 0, & \text{otherwise} \end{cases} \qquad \beta_{2m+1\,j} = \delta_{2m+1,j}. \tag{4.8}$$

Here we will use these explicit values in the general formulas of Section 2 to compute the matrix elements in (2.24). In particular, we will give their asymptotic form in the

neighbourhood of the soft edge, which is specified by introducing the scaled variables X and t according to

$$x = \sqrt{2N} + \frac{X}{2^{1/2}N^{1/6}}, \qquad \tau = \frac{t}{N^{1/3}}. \tag{4.9}$$

The corresponding scaled distribution function (2.8), to be denoted $\rho^{\rm soft}_{(n+m)}$, is given by

$$\begin{aligned}
&\rho^{\rm soft}_{(n+m)}(X_1^{(1)},\dots,X_n^{(1)};t^{(1)};X_1^{(2)},\dots,X_m^{(2)};t^{(2)}) \\
&\quad = \lim_{N\to\infty}\Big(\frac{1}{2^{1/2}N^{1/6}}\Big)^{n+m}\rho_{(n+m)}\Big(\sqrt{2N}+\frac{X_1^{(1)}}{2^{1/2}N^{1/6}},\dots, \\
&\quad \sqrt{2N}+\frac{X_n^{(1)}}{2^{1/2}N^{1/6}};\frac{t^{(1)}}{N^{1/3}};\sqrt{2N}+\frac{X_1^{(2)}}{2^{1/2}N^{1/6}},\dots,\sqrt{2N}+\frac{X_m^{(2)}}{2^{1/2}N^{1/6}};\frac{t^{(2)}}{N^{1/3}}\Big)
\end{aligned} \tag{4.10}$$

Analogous to (3.12)–(3.15), the matrix elements of $\rho^{\rm soft}_{(n+m)}$ are computed from the formulas

$$\begin{aligned}
&S_1^{\rm soft}(X,Y;t_X,t_Y) \\
&\quad = \lim_{N\to\infty}\frac{e^{-N^{2/3}(t_Y-t_X)}}{2^{1/2}N^{1/6}}S\Big(\sqrt{2N}+\frac{X}{2^{1/2}N^{1/6}},\sqrt{2N}+\frac{Y}{2^{1/2}N^{1/6}};\frac{t_X}{N^{1/3}},\frac{t_Y}{N^{1/3}}\Big)
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
&\tilde{S}_1^{\rm soft}(X,Y;t_X,t_Y) \\
&\quad = \lim_{N\to\infty}\frac{e^{-N^{2/3}(t_Y-t_X)}}{2^{1/2}N^{1/6}}\tilde{S}\Big(\sqrt{2N}+\frac{X}{2^{1/2}N^{1/6}},\sqrt{2N}+\frac{Y}{2^{1/2}N^{1/6}};\frac{t_X}{N^{1/3}},\frac{t_Y}{N^{1/3}}\Big)
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
&\tilde{I}_1^{\rm soft}(X,Y;t_X,t_Y) \\
&\quad = \lim_{N\to\infty}e^{N^{2/3}(t_Y+t_X)}\tilde{I}\Big(\sqrt{2N}+\frac{X}{2^{1/2}N^{1/6}},\sqrt{2N}+\frac{Y}{2^{1/2}N^{1/6}};\frac{t_X}{N^{1/3}},\frac{t_Y}{N^{1/3}}\Big)
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&D_1^{\rm soft}(X,Y;t_X,t_Y) \\
&\quad = \lim_{N\to\infty}\frac{e^{-N^{2/3}(t_Y+t_X)}}{2N^{1/3}}D\Big(\sqrt{2N}+\frac{X}{2^{1/2}N^{1/6}},\sqrt{2N}+\frac{Y}{2^{1/2}N^{1/6}};\frac{t_X}{N^{1/3}},\frac{t_Y}{N^{1/3}}\Big).
\end{aligned} \tag{4.14}$$

Here the scale factors in addition to $1/2^{1/2}N^{1/6}$ do not effect the value of (2.24) because they result from a transformation of the form (3.16).

4.1 $\beta = 1$ initial conditions

We see from (4.7) that for N even and $p \ge N$

$$\beta_{pl} = \beta_{pN-1}\,\beta_{N-1l}\,. \tag{4.15}$$

Substituting in the formula (2.29) for S_2 and using (2.17) and (2.18) we thus have

$$\begin{aligned}
S_2(x,y;\tau_x,\tau_y) \;&=\; \sqrt{w(y)}\Big(\frac{\Phi_{N-2}(x;\tau_x)}{r_{N/2-1}(0)}e^{\gamma_{N-2}\tau_x} - \sqrt{w(x)}\frac{C_{N-1}(x)e^{-\gamma_{N-1}\tau_x}}{h_{N-1}}\Big) \\
&\quad \times C_{N-1}(y)e^{\gamma_{N-1}\tau_y}.
\end{aligned} \tag{4.16}$$

To compute the scaled limit we make use of the asymptotic expansion [13]

$$e^{-x^2/2}H_n(x) = \pi^{-3/4}2^{n/2+1/4}(n!)^{1/2}n^{-1/12}\Big(\pi \mathrm{Ai}(-u) + \mathrm{O}(n^{-2/3})\Big) \tag{4.17}$$

where $x = (2n)^{1/2} - u/2^{1/2}n^{1/6}$. This gives

$$\sqrt{w(y)}C_{N-1}(y)\Big|_{y=(2N)^{1/2}+Y/2^{1/2}N^{1/6}} \sim \pi^{1/4}2^{-(N-1)/2+1/4}(N-1)!^{1/2}N^{-1/12}\mathrm{Ai}(Y). \tag{4.18}$$

Also, from (2.18), (4.1) and (4.7)

$$\begin{aligned} f(x;\tau_x) &:= \frac{\Phi_{N-2}(x;\tau_x)}{r_{N/2-1}(0)}e^{\gamma_{N-2}\tau_x} - \sqrt{w(x)}\frac{C_{N-1}(x)e^{-\gamma_{N-1}\tau_x}}{h_{N-1}} \\ &= \frac{e^{-x^2/2}}{(N/2-1)!}e^{-N\tau_x}\sum_{p=0}^{\infty}\frac{(N/2+p)!}{\sqrt{\pi}(N+2p+1)!}H_{N+2p+1}(x)e^{-(2p+3/2)\tau_x}. \end{aligned}$$

Noting that for large N

$$\frac{(N/2+p)!}{((N+2p+1)!)^{1/2}}\frac{((N-1)!)^{1/2}}{(N/2-1)!} \sim 2^{-(p+1)}$$

and making use of (4.18) with $n = N+2p+1$, $-u \sim X - 2p/N^{1/3}$ shows

$$\begin{aligned} &((N-1)!)^{1/2}f\Big((2N)^{1/2} + \frac{X}{2^{1/2}N^{1/6}};\frac{t_X}{N^{1/3}}\Big) \\ &\quad \sim e^{-N^{2/3}t}2^{(N-1)/2}2^{1/4}\pi^{-1/4}N^{-1/12}\frac{N^{1/3}}{2}\int_0^\infty \mathrm{Ai}(X-v)e^{-vt_X}\,dv. \end{aligned} \tag{4.19}$$

Hence

$$\begin{aligned} &S_2\Big((2N)^{1/2} + \frac{X}{2^{1/2}N^{1/6}},(2N)^{1/2} + \frac{Y}{2^{1/2}N^{1/6}};\frac{t_X}{N^{1/3}},\frac{t_Y}{N^{1/3}}\Big) \\ &\quad \sim e^{N^{2/3}(t_Y-t_X)}2^{-1/2}N^{1/6}\mathrm{Ai}(Y)\int_0^\infty \mathrm{Ai}(X-v)e^{-vt_X}\,dv. \end{aligned} \tag{4.20}$$

Now consider S_1 as defined by (2.28). Substituting (4.1) we see that the asymptotics can be determined by writing the sum so that the sum index p is replaced by $N-p$, and then using Stirling's formula and (4.18). We thus find

$$\begin{aligned} &S_1\Big((2N)^{1/2} + \frac{X}{2^{1/2}N^{1/6}},(2N)^{1/2} + \frac{Y}{2^{1/2}N^{1/6}};\frac{t_X}{N^{1/3}},\frac{t_Y}{N^{1/3}}\Big) \\ &\quad \sim e^{N^{2/3}(t_Y-t_X)}2^{1/2}N^{1/6}\int_0^\infty \mathrm{Ai}(X+v)\mathrm{Ai}(Y+v)e^{-v(t_Y-t_X)}\,dv. \end{aligned} \tag{4.21}$$

Adding together (4.20) and (4.21) and substituting for S in (4.11) we thus have

$$S_1^{\mathrm{soft}}(X,Y;t_X,t_Y) = \int_0^\infty \mathrm{Ai}(X+v)\mathrm{Ai}(Y+v)e^{-v(t_Y-t_X)}\,dv + \frac{1}{2}\mathrm{Ai}(Y)\int_0^\infty \mathrm{Ai}(X-v)e^{-vt_X}\,dv. \tag{4.22}$$

Asymptotics of $\tilde{I}(x,y;\tau_x,\tau_y)$

We know from (2.21) that $\tilde{I}$ is defined in terms of $\{\Phi_l(\cdot;\tau)\}$. The asymptotics of Φ_{2k-2} is calculated in an analogous fashion to the expansion (4.19). Thus after noting

$$\Phi_{N+2k-2}(x;\tau)e^{\gamma_{N+2k-2}\tau} = \frac{r_{N/2+k-1}(0)}{(N/2+k-1)!}e^{-N\tau}\sum_{m=k-1}^{\infty}\frac{H_{2m+N+1}(x)(N/2+m)!}{\pi^{1/2}(N+2m+1)!}e^{-(2m+3/2)\tau}$$

we find

$$\Phi_{N+2k-2}\Big((2N)^{1/2}+\frac{X}{2^{1/2}N^{1/6}};\frac{t_X}{N^{1/3}}\Big)e^{\gamma_{N+2k-2}t_X/N^{1/3}} \sim \frac{r_{N/2+k-1}(0)}{((N-1)!)^{1/2}}\frac{(N/2-1)!}{(N/2+k-1)!} \times e^{-N^{2/3}t_X}2^{(N-1)/2}2^{1/4}\pi^{-1/4}N^{-1/12}\frac{N^{1/3}}{2}\int_{2k/N^{1/3}}^{\infty}\mathrm{Ai}(X-v)e^{-vt_X}\,dv \quad (4.23)$$

Consider now Φ_{2k-1}. We note from (2.18), (4.1) and (4.7) that

$$\Phi_{N+2k-1}(y;\tau)e^{\gamma_{N+2k-1}\tau} = -\sqrt{w(y)}r_{N/2+k-1}(0)\frac{H_{N+2k-2}(y)}{\pi^{1/2}(N+2k-2)!}e^{-N\tau}e^{-(2k-3/2)\tau}$$

so use of (4.17) gives

$$\Phi_{N+2k-1}\Big((2N)^{1/2}+\frac{Y}{2^{1/2}N^{1/6}};\frac{t_Y}{N^{1/3}}\Big)e^{\gamma_{N+2k-1}\tau_Y/N^{1/3}} \sim -r_{N/2+k-1}(0)e^{-N^{2/3}t_Y} \times e^{-2kt_Y/N^{1/3}}\pi^{-1/4}2^{(N+2k-2)/2+1/4}((N+2k-2)!)^{1/2}N^{-1/12}\mathrm{Ai}(Y-\frac{2k}{N^{1/3}}). \quad (4.24)$$

Multiplying (4.23) and (4.24) together, substituting in (2.21) and making use of the explicit value of $r_m(0)$ from (4.4), we then find

$$\tilde{I}_1^{\rm soft}(X,Y;t_X,t_Y) = \int_0^\infty ds\,\mathrm{Ai}(Y-s)e^{-st_Y}\int_s^\infty \mathrm{Ai}(X-v)e^{-vt_x}dv - (X\leftrightarrow Y) \quad (4.25)$$

Asymptotics of $D(x,y;\tau_x,\tau_y)$

Here the asymptotics of $\{R_m(\cdot;\tau)\}$ are required. Now, from (2.14) and (4.4) we have

$$R_{N-2l-2}(x;\tau)e^{\gamma_{N-2l-2}\tau} = 2^{-(N-2l-2)}H_{N-2l-2}(x)e^{\gamma_{N-2l-2}\tau}.$$

Use of (4.17) then gives

$$\sqrt{w(x)}R_{N-2l-2}(x;\tau)e^{\gamma_{N-2l-2}\tau}\Big|_{\substack{x=(2N)^{1/2}+X/2^{1/2}N^{1/6}\\ \tau=t_X/N^{1/3}}} \sim e^{N^{2/3}t_X} \times e^{-2lt_X/N^{1/3}}\pi^{1/4}2^{-(N-2l-2)/2}2^{1/4}((N-2l-2)!)^{1/2}N^{-1/12}\mathrm{Ai}\Big(X+\frac{2l}{N^{1/3}}\Big). \quad (4.26)$$

Furthermore

$$\begin{aligned} R_{2m+1}(x;\tau)e^{\gamma_{2m+1}\tau} &= 2^{-(2m+1)}H_{2m+1}(x)e^{\gamma_{2m+1}\tau} - m2^{-(2m-1)}H_{2m-1}(x)e^{\gamma_{2m-1}\tau} \\ &= -e^{\gamma_{2m+1}\tau}e^{x^2/2}\frac{d}{dx}\Big(e^{-x^2/2}2^{-2m}H_{2m}(x)\Big) \\ &\quad -m2^{-(2m-1)}H_{2m-1}(x)e^{\gamma_{2m-1}\tau}(1-e^{2\tau}), \end{aligned} \quad (4.27)$$

and use of (4.17) gives for the asymptotics

$$\sqrt{w(x)}R_{N-2l-1}(y;\tau)e^{\gamma_{N-2l-1}\tau}\Big|_{\substack{y=(2N)^{1/2}+Y/2^{1/2}N^{1/6}\\ \tau=t_Y/N^{1/3}}} \sim e^{N^{2/3}t_Y}$$

$$\times\pi^{1/4}2^{-(N-2l-2)/2}2^{3/4}((N-2l-2)!)^{1/2}N^{1/12}\frac{d}{ds}\Big(e^{-st_Y}\mathrm{Ai}(Y+s)\Big), \qquad (4.28)$$

$s := 2l/N$. Multiplying together (4.26) and (4.28), dividing by $r_{(N-2l-2)/2}$ and summing over l (which is of the form of a Riemann integral) we thus obtain

$$D_1^{\mathrm{soft}}(X,Y;t_X,t_Y) = \frac{1}{4}\int_0^\infty ds\, e^{-st_X}\mathrm{Ai}(X+s)\frac{d}{ds}\Big(e^{-st_Y}\mathrm{Ai}(Y+s)\Big) - (X\leftrightarrow Y) \quad (4.29)$$

Asymptotics of $g(x,y;\tau_y-\tau_x)$

According to (2.3) and (2.4)

$$g(x,y;\tau) = e^{-(x^2+y^2)/2}\sum_{j=0}^\infty \frac{H_j(x)H_j(y)e^{-(j+1/2)\tau}}{\pi^{1/2}2^j j!} \qquad (4.30)$$

This summation can be evaluated in closed form [11] giving

$$g(x,y;\tau) = e^{-(x^2+y^2)/2}\frac{e^{-\tau/2}}{(1-e^{-2\tau})^{1/2}}\exp\Big(-\frac{e^{-2\tau}}{1-e^{-2\tau}}(x-y)^2\Big)\exp\Big(\frac{2e^{-\tau}}{1-e^{-2\tau}}xy\Big)$$

Hence

$$g\Big((2N)^{1/2}+\frac{X}{\sqrt{2}N^{1/6}},(2N)^{1/2}+\frac{Y}{\sqrt{2}N^{1/6}};\frac{t}{N^{1/3}}\Big) \sim \frac{N^{1/6}}{(2t)^{1/2}}e^{-tN^{2/3}}e^{-(X-Y)^2/4t-t(X+Y)/2}.$$

Alternatively, proceeding as in the derivation of (4.21), we can deduce from (4.30) that

$$g\Big((2N)^{1/2}+\frac{X}{\sqrt{2}N^{1/6}},(2N)^{1/2}+\frac{Y}{\sqrt{2}N^{1/6}};\frac{t}{N^{1/3}}\Big)$$
$$\sim e^{-N^{2/3}t}2^{1/2}N^{1/6}\int_{-\infty}^\infty \mathrm{Ai}(X+v)\mathrm{Ai}(Y+v)e^{vt}\,dv \qquad (4.31)$$

Dividing by $e^{-N^{2/3}t}2^{1/2}N^{1/6}$, replacing t by t_X-t_Y and subtracting from (4.22) shows

$$\begin{aligned}\tilde{S}_1^{\mathrm{soft}}(X,Y;t_X,t_Y) &= -\int_{-\infty}^0 \mathrm{Ai}(X+v)\mathrm{Ai}(Y+v)e^{-v(t_Y-t_X)}\,dv\\ &\quad+\frac{1}{2}\mathrm{Ai}(Y)\int_0^\infty \mathrm{Ai}(X-v)e^{-vt_X}\,dv. \end{aligned}\qquad (4.32)$$

4.2 $\beta=4$ initial conditions

Here we want to again compute the scaled distribution (4.10). As for $\beta=4$ initial conditions in the Laguerre ensemble, we must modify the scale factor in (4.13) and (4.14) so that

$$\tilde{I}_4^{\mathrm{soft}}(X,Y;t_X,t_Y)$$
$$=\lim_{N\to\infty}\frac{e^{N^{2/3}(t_X+t_Y)}}{2N^{1/3}}\tilde{I}\Big((2N)^{1/2}+\frac{X}{\sqrt{2}N^{1/6}},(2N)^{1/2}+\frac{Y}{\sqrt{2}N^{1/6}};\frac{t}{N^{1/3}}\Big) \quad (4.33)$$

$$
\begin{aligned}
&D_4^{\rm soft}(X,Y;t_X,t_Y)\\
&\quad = \lim_{N\to\infty} e^{-N^{2/3}(t_X+t_Y)} D\Big((2N)^{1/2}+\frac{X}{\sqrt{2}N^{1/6}},(2N)^{1/2}+\frac{Y}{\sqrt{2}N^{1/6}};\frac{t}{N^{1/3}}\Big)
\end{aligned} \tag{4.34}
$$

Substituting (4.8) in (2.29) shows that in terms of $\{C_l(x)\}$, S_2 only consists of a single term, which reads explicitly

$$
S_2(x,y;\tau_x,\tau_y) = \sqrt{w(x)w(y)}\frac{C_N(x)e^{-\gamma_N\tau_x}}{h_N}\Big(-\frac{N}{2}\Big)R_{N-2}(y;\tau_y)e^{\gamma_{N-2}\tau_y}. \tag{4.35}
$$

The explicit form of $R_{N-2}(y;\tau_y)e^{\gamma_{N-2}\tau_y}$ is given by (4.6). This expression however does not give a Riemann sum after substituting the asymptotic form of the integrand. Instead, we make use of the identity

$$
\sum_{l=0}^m \frac{m!}{l!}2^{-2l}H_{2l}(x)c^{-l} = -\alpha(4c)^{-m}e^{\alpha x^2}\int_{-\infty}^x e^{-\alpha s^2}H_{2m+1}(s)\,ds, \tag{4.36}
$$

$c=(1-\alpha)/\alpha$. This can be verified by multiplying both sides by $e^{-\alpha x^2}$ and differentiating (we also verify that both sides have the same $x\to\infty$ behaviour). Choosing $c=e^{-2\tau}$ and comparing with (4.6) we see that

$$
\begin{aligned}
R_{2m}(y;\tau)e^{\gamma_{2m}\tau} &= -\frac{e^{\tau/2}}{1+e^{-2\tau}}(2e^{-\tau})^{-2m}e^{y^2/(1+e^{-2\tau})}\int_{-\infty}^y e^{-s^2/(1+e^{-2\tau})}H_{2m+1}(s)\,ds\\
&= \frac{e^{\tau/2}}{1+e^{-2\tau}}(2e^{-\tau})^{-2m}e^{y^2/(1+e^{-2\tau})}\int_y^\infty e^{-s^2/(1+e^{-2\tau})}H_{2m+1}(s)\,ds
\end{aligned} \tag{4.37}
$$

where to obtain the second equality the fact that the integrand is odd has been used.

With $x=(2N)^{1/2}+X/2^{1/2}N^{1/6}$, making the change of variables $s\mapsto(2N)^{1/2}+s$ and using the asymptotic expansion (4.17) we see that

$$
\begin{aligned}
\sqrt{w(y)}R_{N-2m}\Big((2N)^{1/2}+\frac{Y}{2^{1/2}N^{1/6}};\frac{t}{N^{1/3}}\Big)e^{\gamma_{N-2m}t/N^{1/3}} &\sim 2^{-3/4}N^{-1/4}\pi^{1/4}2^{-(N-2m)/2}\\
\times((N-2m+1)!)^{1/2}e^{N^{2/3}t}e^{Yt}\int_{Y+u}^\infty e^{-st}\mathrm{Ai}(s)\,ds,&\quad u:=2m/N^{1/3}
\end{aligned} \tag{4.38}
$$

For the x-dependent factor in (4.35), the asymptotic form of $\sqrt{w(x)}C_N(x)$ is given by (4.18) with $N-1$ replaced by N. Making use of the explicit value of h_N, we thus find

$$
\begin{aligned}
&S_2\Big((2N)^{1/2}+\frac{X}{2^{1/2}N^{1/6}},(2N)^{1/2}+\frac{Y}{2^{1/2}N^{1/6}};\frac{t_X}{N^{1/3}},\frac{t_Y}{N^{1/3}}\Big)\\
&\quad\sim -\frac{N^{1/6}}{2^{1/2}}e^{-N^{2/3}(t_X-t_Y)}\mathrm{Ai}(X)e^{Yt}\int_Y^\infty e^{-st}\mathrm{Ai}(s)\,ds.
\end{aligned} \tag{4.39}
$$

The asymptotic form of the quantity S_1 as defined by (2.28) is given by (4.21), since S_1 is the same for $\beta=1$ and $\beta=4$ initial conditions. Hence

$$
S_4^{\rm soft}(X,Y;t_X,t_Y) = \int_0^\infty \mathrm{Ai}(X+v)\mathrm{Ai}(Y+v)e^{-v(t_Y-t_X)}\,dv \tag{4.40}
$$

$$
-\frac{1}{2}\mathrm{Ai}(X)e^{Yt}\int_Y^\infty e^{-st}\mathrm{Ai}(s)\,ds \tag{4.41}
$$

Also, subtracting (4.31) divided by $e^{-N^{2/3}t}2^{1/2}N^{1/6}$ (and with $t \mapsto t_X - t_Y$) gives

$$\begin{aligned} \tilde{S}_4^{\rm soft}(X,Y;t_X,t_Y) &= -\int_{-\infty}^0 \mathrm{Ai}(X+v)\mathrm{Ai}(Y+v)e^{-v(t_Y-t_X)}\,dv \\ & \quad -\frac{1}{2}\mathrm{Ai}(X)e^{Yt}\int_Y^\infty e^{-st}\mathrm{Ai}(s)\,ds \end{aligned} \tag{4.42}$$

Asymptotics of $\tilde{I}(x,y;\tau_x,\tau_y)$

First note from (2.18), (4.1), (4.6) and (4.8) that

$$\begin{aligned} &\Phi_{2k-1}(x;\tau)e^{\gamma_{2k-1}\tau} \\ &\quad = \sqrt{w(x)}e^{-\gamma_{2k-2}\tau}2^{-2(k-1)}\Big(\frac{1}{2}H_{2k}(x)e^{-2\tau} - (2k-1)H_{2k-2}(x)\Big) \\ &\quad = -e^{-\gamma_{2k-2}\tau}2^{-2(k-1)}\Big(\frac{d}{dx}\Big(e^{-x^2/2}H_{2k-1}(x)\Big) - \frac{\sqrt{w(x)}}{2}H_{2k}(x)(e^{-2\tau}-1)\Big) \end{aligned}$$

(c.f. (4.27)). Use of (4.17) gives for the asymptotics

$$\begin{aligned} &\Phi_{N-2k-1}\Big((2N)^{1/2} + \frac{X}{2^{1/2}N^{1/6}};\frac{t}{N^{1/3}}\Big)e^{\gamma_{N-2k-1}t/N^{1/3}} \sim \\ &\quad -e^{-N^{2/3}t_X}\pi^{1/4}2^{7/4}2^{-(N-2k-1)/2}(N-2k-1)!^{1/2}N^{1/12}\frac{d}{ds}\Big(e^{st_X}\mathrm{Ai}(X+s)\Big), \end{aligned} \tag{4.43}$$

$s := 2k/N$. Similarly

$$\frac{1}{r_{k-1}(0)}\Phi_{2k-2}(y;\tau)e^{\gamma_{2k-2}\tau} = \sqrt{w(y)}\frac{H_{2k-1}(y)e^{-\gamma_{2k-1}\tau}}{\pi^{1/2}(2k-1)!}$$

and so

$$\begin{aligned} &\frac{1}{r_{N/2-k-1}(0)}\Phi_{N-2k-2}\Big((2N)^{1/2} + \frac{Y}{2^{1/2}N^{1/6}};\frac{t}{N^{1/3}}\Big)e^{\gamma_{N-2k-2}t/N^{1/3}} \sim \\ &\quad e^{-N^{2/3}t_Y}e^{st_Y}\pi^{-1/4}2^{(N-2k-1)/2+1/4}(N-2k-1)!^{-1/2}N^{-1/12}\mathrm{Ai}(Y+s). \end{aligned} \tag{4.44}$$

Multiplying together (4.43) and (4.44) and summing according to (2.21) we see that

$$\tilde{I}_4^{\rm soft}(X,Y;t_X,t_Y) = \int_0^\infty ds\, e^{st_Y}\mathrm{Ai}(Y+s)\frac{d}{ds}\Big(e^{st_X}\mathrm{Ai}(X+s)\Big) - (X \leftrightarrow Y) \tag{4.45}$$

Asymptotics of $D(x,y;\tau_x,\tau_y)$

Here we require the asymptotics of $R_{2m}(\cdot;\tau)$ and $R_{2m+1}(\cdot;\tau)$. The former is given by (4.38). For the latter, we note from (4.6) that

$$\frac{1}{r_m(0)}R_{2m+1}(x;\tau)e^{\gamma_{2m+1}\tau} = \frac{H_{2m+1}(y)}{2\sqrt{\pi}\Gamma(2m+2)}e^{\gamma_{2m+2}\tau}$$

and thus according to (4.17)

$$\frac{\sqrt{w(x)}}{r_{N/2-m}(0)} R_{N-2m+1}(x;\tau) e^{\gamma_{N-2m+1}\tau}\Big|_{\substack{x=(2N)^{1/2}+X/2^{1/2}N^{1/6} \\ \tau=t/N^{1/3}}} \tag{4.46}$$

$$\sim \frac{e^{N\tau_x}}{2\sqrt{\pi}} \frac{e^{-ut_X}}{(\Gamma(N-2m+2))^{1/2}} \pi^{1/4} 2^{(N-2m+1)/2+1/4} N^{-1/12} \mathrm{Ai}(X+u). \tag{4.47}$$

Multiplying (4.46) and (4.38) according to (2.22) we see that

$$D(X,Y;t_X,t_Y) = \frac{1}{4} e^{N^{2/3}(t_X+t_Y)} \int_0^\infty e^{-ut_Y} \mathrm{Ai}(Y+u) e^{Xt_X} \int_{X+u}^\infty e^{-st} \mathrm{Ai}(s)\, ds - (X \leftrightarrow Y). \tag{4.48}$$

5 Discussion

5.1 Connection with $\beta = 2$ initial conditions

We have calculated exact formulas for the dynamical distribution function (2.8) in the case of $\beta = 1$ and $\beta = 4$ initial conditions, with the final state corresponding to a $\beta = 2$ equilibrium. If in these formulas we take the limit $t_X, t_Y \to \infty$ with $t_X - t_Y$ fixed, then we would expect no memory of the initial state to be retained, but rather the distribution functions corresponding to a $\beta = 2$ initial condition (i.e. the equilibrium state) to result.

Consider first the hard edge. Inspection of (3.31), (3.37), (3.49) and (3.50) shows that in this limit

$$\begin{aligned} S_1^{\rm hard}(X,Y;t_X,t_Y) &\sim S_4^{\rm hard}(X,Y,;t_X,t_Y) \\ &\sim S_2^{\rm hard}(X,Y,;t_X,t_Y) := \frac{1}{4}\int_0^1 J_a\left(\sqrt{uX}\right) J_a\left(\sqrt{uY}\right) e^{-u(t_X-t_Y)}\, du\,, \end{aligned} \tag{5.1}$$

$$\begin{aligned} \tilde{S}_1^{\rm hard}(X,Y;t_X,t_Y) &\sim \tilde{S}_4^{\rm hard}(X,Y,;t_X,t_Y) \\ &\sim \tilde{S}_2^{\rm hard}(X,Y,;t_X,t_Y) := -\frac{1}{4}\int_1^\infty J_a\left(\sqrt{uX}\right) J_a\left(\sqrt{uY}\right) e^{-u(t_X-t_Y)}\, du\,, \end{aligned} \tag{5.2}$$

while from (3.33), (3.54), (3.38) and (3.57) we see that

$$e^{(t_X+t_Y)}\, \tilde{I}_1^{\rm hard}(X,Y;t_X,t_Y) \sim e^{(t_X+t_Y)}\, \tilde{I}_4^{\rm hard}(X,Y;t_X,t_Y) \sim 0\,, \tag{5.3}$$

$$e^{-(t_X+t_Y)}\, D_1^{\rm hard}(X,Y;t_X,t_Y) \sim e^{-(t_X+t_Y)}\, D_4^{\rm hard}(X,Y;t_X,t_Y) \sim 0\,. \tag{5.4}$$

Substituting these values in (2.24) and then using (2.26), and the formula

$$\mathrm{Pf(ZX)} = (\mathrm{det(ZX)})^{1/2},$$

we see by interchanges of particular rows and columns that ZX is equivalent to a block matrix of the form

$$\begin{pmatrix} 0_{n+m} & A \\ -A & 0_{n+m} \end{pmatrix},$$

where

$$A = \left[\begin{array}{cc} \left[S_2^{\text{hard}}\left(X_j^{(1)}, X_k^{(1)}; t_1, t_1\right)\right]_{n\times n} & \left[\tilde{S}_2^{\text{hard}}\left(X_j^{(1)}, X_k^{(2)}; t_1, t_2\right)\right]_{n\times m} \\ \left[\tilde{S}_2^{\text{hard}}\left(X_j^{(2)}, X_k^{(1)}; t_2, t_1\right)\right]_{m\times n} & \left[S_2^{\text{hard}}\left(X_j^{(2)}, X_k^{(2)}; t_2, t_2\right)\right]_{m\times m} \end{array} \right]. \tag{5.5}$$

Thus the limiting form of the dynamical distribution is given by

$$\rho_{(n+m)}\left(x_1^{(1)}, \dots, x_n^{(1)}; \tau_1; x_1^{(2)}, \dots, x_m^{(2)}; \tau_2\right) = \det \mathrm{A}\,. \tag{5.6}$$

The analysis in the soft edge case is very similar. The formulas (5.3) and (5.4) again apply, as do the formulas (5.1) and (5.2) with S_2^{hard}, $\tilde{S}_2^{\text{hard}}$ replaced by

$$S_2^{\text{soft}}(X, Y; t_X, t_Y) := \int_0^\infty \mathrm{Ai}(X+v)\mathrm{Ai}(Y+v)e^{-v(t_Y-t_X)}dv\,, \tag{5.7}$$

$$\tilde{S}_2^{\text{soft}}(X, Y; t_X, t_Y) := -\int_{-\infty}^0 \mathrm{Ai}(X+v)\mathrm{Ai}(Y+v)e^{-v(t_Y-t_X)}dv\,. \tag{5.8}$$

We remark that in the case $n = m = 1$, the resulting formula for $\rho_{(n+m)}$ implied by (5.6) and (5.5) in both the hard and soft edge theories agrees with that computed by Mâcedo [14, 15].

5.2 Connection between the soft and hard edges

For the hard edge distribution function of the Laguerre ensemble at $\beta = 2$ and $\tau = 0$, it is known that by taking the limit $a \to \infty$ and introducing new scaled coordinates, the distribution function of the soft edge of the Gaussian ensemble results [16]. Specifically, the new scaled variable x is related to the scaled variable X of the hard edge by

$$X = a^2 - 2\,a\,(a/2)^{1/3}\,x \tag{5.9}$$

and the distribution functions are related by

$$\lim_{a\to\infty}\left(-2a(a/2)^{1/3}\right)^n \rho_{(n)}^{\text{hard}}\left(a^2 - 2a(a/2)^{1/3}x_1, \dots, a^2 - 2a(a/2)^{1/3}x_n\right) = \rho_{(n)}^{\text{soft}}\left(x_1, \dots, x_n\right)\,. \tag{5.10}$$

As noted in [16, 17] this property should persist for general β. Further, with appropriate scaling of τ, the result would be expected to generalize to the dynamical distributions.

Let us first check this latter point for the dynamical distribution with a $\beta = 2$ initial condition (5.6). For this purpose we require the asymptotic expansion [18]

$$J_a(x) \sim \left(\frac{2}{x}\right)^{1/3} \mathrm{Ai}\left(\frac{2^{1/3}(a-x)}{x^{1/3}}\right) \tag{5.11}$$

valid for a and x large such that the argument of the Airy function is of order unity. Changing variables $u = 1 - w(2/a)^{2/3}$ in the integrals of (5.1) and (5.2) and introducing the scaled coordinates (5.9), application of (5.11) shows

$$J_a\left(\sqrt{ux}\right) \sim \left(\frac{2}{a}\right)^{1/3} \mathrm{Ai}(x+w)\,. \tag{5.12}$$

Thus if we also scale t according to

$$t_X \mapsto (a/2)^{2/3}\, t_x \tag{5.13}$$

then we see that

$$\begin{aligned}
&\left(-2a(a/2)^{1/3}\right) S_2^{\rm hard}\left(a^2 - 2a(a/2)^{1/3}x, a^2 - 2a(a/2)^{1/3}y; (a/2)^{2/3}t_x, (a/2)^{2/3}t_y\right) \\
&\qquad\qquad \sim e^{-(a/2)^{2/3}(t_x - t_y)} S_2^{\rm soft}(x, y; t_x, t_y)\,, \\
&\left(-2a(a/2)^{1/3}\right) \tilde{S}_2^{\rm hard}\left(a^2 - 2a(a/2)^{1/3}x, a^2 - 2a(a/2)^{1/3}y; (a/2)^{2/3}t_x, (a/2)^{2/3}t_y\right) \\
&\qquad\qquad \sim e^{-(a/2)^{2/3}(t_x - t_y)} S_2^{\rm soft}(x, y; t_x, t_y)\,.
\end{aligned}$$

Since, as seen from (2.24), the factors $e^{-(a/2)^{2/3}(t_x - t_y)}$ do not affect the value of $\rho_{(n+m)}$, we thus have the dynamical extension of (5.10),

$$\begin{aligned}
\lim_{a\to\infty} \left(-2a(a/2)^{1/3}\right)^{n+m} \rho_{(n+m)}^{\rm hard}\Big(&a^2 - 2a(a/2)^{1/3}x_1, \dots, a^2 - 2a(a/2)^{1/3}x_n; (a/2)^{2/3}t_x; \\
&a^2 - 2a(a/2)^{1/3}y_1, \dots, a^2 - 2a(a/2)^{1/3}y_m; (a/2)^{2/3}t_y\Big) \\
&= \rho_{(n+m)}^{\rm hard}(x_1, \dots, x_n; t_x; y_1, \dots, y_m; t_y)\,.
\end{aligned} \tag{5.14}$$

Consider next $\beta = 1$ initial conditions. Inspection of (3.31), (3.37), (3.33) and (3.38) shows that the analysis used above for $\beta = 2$ initial conditions again suffices to establish the connection formula. For example, consider the formula (3.33) for $\tilde{I}_1^{\rm hard}$. For large a we see that

$$\begin{aligned}
&\tilde{I}_1^{\rm hard}(X, Y; t_X, t_Y) \\
&\quad\sim\ \frac{a^2}{4}\int_1^\infty s^{-1+a/2} J_a\left(\sqrt{sY}\right) e^{-st_Y} \int_s^\infty \frac{e^{-ut_X}}{u} J_a\left(\sqrt{uX}\right) du - (X \leftrightarrow Y).
\end{aligned} \tag{5.15}$$

Use of the asymptotic expansion (5.12) (after making the appropriate change of variables in the integrals) shows

$$\begin{aligned}
&\left(-2a(a/2)^{1/3}\right) \tilde{I}_1^{\rm hard}\left(a^2 - 2a(a/2)^{1/3}x, a^2 - 2a(a/2)^{1/3}y; (a/2)^{2/3}t_x, (a/2)^{2/3}t_y\right) \\
&\qquad\qquad \sim e^{-(a/2)^{2/3}(t_x + t_y)} \tilde{I}_1^{\rm soft}(x, y; t_x, t_y)\,.
\end{aligned}$$

Similar formulas hold connecting $S_1^{\rm hard}$, $\tilde{S}_1^{\rm hard}$ and $D_1^{\rm hard}$ to $S_1^{\rm soft}$, $\tilde{S}_1^{\rm soft}$ and $D_1^{\rm soft}$ respectively, thus showing the formula (5.14) holds for $\beta = 1$ initial conditions. Repeating the analysis for $\beta = 4$ initial conditions leads to the same conclusion.

5.3 Connection with the bulk

It is known [16] that after appropriate scaling of the variables, the static distributions at the hard and soft edges tend to the bulk distributions at large distances from the edge of the system. The required scaling is

$$X \mapsto \alpha + 2\pi\sqrt{\alpha}\,\rho x\,, \qquad X \mapsto -\left(\alpha + \pi\rho x/\sqrt{\alpha}\right)\,, \tag{5.16}$$

with $\alpha \to \infty$, for the hard and soft edges respectively. Here we will show that this feature persists for the dynamical distributions.

Consider first the hard edge with $\beta = 2$ initial conditions. Changing variables $u \mapsto u^2$ and making use of the asymptotic expansion

$$J_a(x) \underset{x\to\infty}{\sim} \left(\frac{2}{\pi x}\right)^{1/2} \cos(x - \pi a/2 - \pi/4) \tag{5.17}$$

in the formulas (5.1) and (5.2), we see that

$$\begin{aligned} &\lim_{\alpha\to\infty} 2\pi\sqrt{\alpha}\,\rho\, S_2^{\rm hard}\left(\alpha + 2\pi\sqrt{\alpha}\,\rho x, \alpha + 2\pi\sqrt{\alpha}\,\rho y; t_x, t_y\right) \\ &\qquad\qquad \sim \quad \rho \int_0^1 \cos(\pi u\rho(x-y))\, e^{-u^2(t_x-t_y)}\, du\,, \qquad (5.18) \\ &\lim_{\alpha\to\infty} 2\pi\sqrt{\alpha}\,\rho\, \tilde{S}_2^{\rm hard}\left(\alpha + 2\pi\sqrt{\alpha}\,\rho x, \alpha + 2\pi\sqrt{\alpha}\,\rho y; t_x, t_y\right) \\ &\qquad\qquad = \quad -\rho \int_1^\infty \cos(\pi u\rho(x-y))\, e^{-u^2(t_x-t_y)}\, du\,. \qquad (5.19) \end{aligned}$$

Substituting these formulas in (5.5) and (5.6) and also making the change of scale $t \mapsto (\pi\rho)^2 t/2$ reclaims the known formula for the bulk dynamical distributions [19].

In fact the asymptotic behaviour (5.18) and (5.19) of $S_\beta^{\rm hard}$ and $\tilde{S}_\beta^{\rm hard}$ for $\beta = 2$ initial conditions persists for $\beta = 1$ and $\beta = 4$ initial conditions. This follows from application of (5.27) in the exact results (3.31), (3.37) and (3.49), (3.50). It remains to consider $\tilde{I}_\beta$ and D_β. For $\beta = 1$ initial conditions application of (5.17) in (3.33) and (3.38) shows

$$\begin{aligned} &\lim_{\alpha\to\infty} 2\pi\rho\, \tilde{I}_1^{\rm hard}\left(\alpha + 2\pi\sqrt{\alpha}\,\rho x, \alpha + 2\pi\sqrt{\alpha}\,\rho y; t_x, t_y\right) \\ &\qquad\qquad = \quad \rho \int_1^\infty du\, \frac{e^{-u^2(t_x+t_y)}}{u} \sin(\pi u\rho(y-x))\,, \qquad (5.20) \\ &\lim_{\alpha\to\infty} 2\pi\rho\alpha\, D_1^{\rm hard}\left(\alpha + 2\pi\sqrt{\alpha}\,\rho x, \alpha + 2\pi\sqrt{\alpha}\,\rho y; t_x, t_y\right) \\ &\qquad\qquad = \quad \rho \int_0^1 du\, u\, e^{u^2(t_x+t_y)} \sin(\pi u\rho(y-x))\,, \qquad (5.21) \end{aligned}$$

while for $\beta = 4$ initial conditions application of (5.17) in (3.54) and (3.57) shows

$$\begin{aligned} &\lim_{\alpha\to\infty} 2\pi\rho\alpha\, \tilde{I}_4^{\rm hard}\left(\alpha + 2\pi\sqrt{\alpha}\,\rho x, \alpha + 2\pi\sqrt{\alpha}\,\rho y; t_x, t_y\right) \\ &\qquad\qquad = \quad \rho \int_1^\infty du\, u\, e^{-u^2(t_x+t_y)} \sin(\pi u\rho(y-x))\,, \qquad (5.22) \\ &\lim_{\alpha\to\infty} 2\pi\rho\, D_4^{\rm hard}\left(\alpha + 2\pi\sqrt{\alpha}\,\rho x, \alpha + 2\pi\sqrt{\alpha}\,\rho y; t_x, t_y\right) \\ &\qquad\qquad = \quad \rho \int_0^1 du\, \frac{e^{u^2(t_x+t_y)}}{u} \sin(\pi u\rho(y-x))\,. \qquad (5.23) \end{aligned}$$

After scaling $t \mapsto (\pi\rho)^2 t/2$, the known formulas for $\rho_{(1+1)}(x, y; t_x, t_y)$ with $\beta = 1$ and $\beta = 4$ initial conditions [20, 21] are reproduced by these results.

In the case of the soft edge connecting to the bulk, similar formulas are found to hold. These formulas are derived from the asymptotic expansion

$$\mathrm{Ai}(-x) \underset{x\to\infty}{\sim} \frac{1}{\pi^{1/2} x^{1/4}} \cos(2x^{3/2}/3 - \pi/4)\,. \tag{5.24}$$

Consider for example $S_2^{\rm soft}$ as specified by (5.7). We first change variables $v \mapsto \alpha(1-u)$ for $0 < v < \alpha$, and replace the Airy functions in the integrand according to (5.24). Noting that

$$\frac{2}{3}\Big\{(\alpha + \pi\rho x/\sqrt{\alpha} - v)^{3/2} - (\alpha + \pi\rho y/\sqrt{\alpha} - v)^{3/2}\Big\} \sim \pi\rho u^{1/2}(x-y)\,,$$

we see that if we also scale t according to

$$t \mapsto t/\alpha \tag{5.25}$$

then $S_2^{\rm soft}$ has the asymptotic form

$$\begin{aligned}\lim_{\alpha\to\infty} (-\pi\rho/\sqrt{\alpha})\, S_2^{\rm soft}(-(\alpha + \pi\rho x/\sqrt{\alpha}), -(\alpha + \pi\rho y/\sqrt{\alpha}); t_x/\alpha, t_y/\alpha) \\ \sim \quad e^{-(t_x-t_y)}\rho \int_0^1 \cos(\pi u\rho(x-y))\, e^{-u^2(t_x-t_y)}\, du\,. \end{aligned} \tag{5.26}$$

Apart from the factor of $e^{-(t_x-t_y)}$ this is the same result as (5.18). A similar calculation shows the asymptotic form of $\tilde{S}_2^{\rm soft}$ is given by (5.19) except for an additional factor of $e^{-(t_x-t_y)}$. This shows that this additional factor cancels from the formula (2.24) for the distribution functions. The final result is that the bulk dynamical distributions are reclaimed from the soft edge results for each of the three distinct initial conditions. (As in the hard edge case a final rescaling $t \mapsto (\pi\rho)^2 t/2$ is required).

5.4 Asymptotic form of $\rho^T_{(1+1)}(X,Y;t)$

The dynamical density-density correlation function between an eigenvalue with value X initially, and an eigenvalue with value Y and parameter τ is defined in terms of the distribution function (2.8) by

$$\rho^T_{(1+1)}(x,y;\tau) = \rho_{(1+1)}(x;0;y;\tau) - \rho_{(1)}(x;0)\,\rho_{(1)}(y;\tau)\,. \tag{5.27}$$

According to (2.24) and (2.26) we have that in the finite system

$$\rho^T_{(1+1)}(x,y;\tau) = -\Big(S(x,y;0,\tau)\,\tilde{S}(y,x;\tau,0) - \tilde{I}(x,y;0,\tau)\,D(x,y;0,\tau)\Big)\,. \tag{5.28}$$

For the Laguerre ensemble at the hard edge we define the corresponding scaled correlation $\rho^{T\,\rm hard}_{(1+1)}$ by

$$\begin{aligned}\rho^{T\,\rm hard}_{(1+1)}(X,Y;t) &= \lim_{N\to\infty}\Big(\frac{1}{4N}\Big)^2 \rho^T_{(1+1)}\Big(\frac{X}{4N},\frac{Y}{4N};\frac{t}{2N}\Big) \\ &= -\Big(S^{\rm hard}_\beta(X,Y;0,t)\,\tilde{S}^{\rm hard}_\beta(Y,X;t,0) - \tilde{I}^{\rm hard}_\beta(X,Y;0,t)\,D^{\rm hard}_\beta(X,Y;0,t)\Big)\,,\end{aligned} \tag{5.29}$$

where $\beta = 1,2$ or 4 depending on the initial conditions, while for the Gaussian ensemble at the soft edge the scaled correlation $\rho^{T\,\rm soft}_{(1+1)}$ is defined by

$$\begin{aligned}\rho^{T\,\rm soft}_{(1+1)}(X,Y;t) &= \lim_{N\to\infty}\Big(\frac{1}{2^{1/2}N^{1/6}}\Big)^2 \rho^T_{(1+1)}\Big(\sqrt{2N} + \frac{X}{2^{1/2}N^{1/6}}, \sqrt{2N} + \frac{Y}{2^{1/2}N^{1/6}}; \frac{t}{N^{1/3}}\Big) \\ &= -\Big(S^{\rm soft}_\beta(X,Y;0,t)\,\tilde{S}^{\rm soft}_\beta(Y,X;t,0) - \tilde{I}^{\rm soft}_\beta(X,Y;0,t)\,D^{\rm soft}_\beta(X,Y;0,t)\Big)\,.\end{aligned} \tag{5.30}$$

The asymptotic form of $\rho^{\rm hard}_{(1+1)}(X,Y;t)$ for large $|X-Y|$ and t is of interest in applications of the parameter dependent Laguerre ensemble to transport in mesoscopic systems [15]. To deduce this asymptotic form, a hydrodynamical approximation to the corresponding Fokker-Planck equation was made which implied

$$\int_0^\infty \int_0^\infty (4XY)\, \rho^{T\,\rm hard}_{(1+1)}\left(X^2,Y^2;t\right)\, \cos(kX)\, \cos(kY)\, dX\, dY \underset{\substack{k\to 0\\ t\to\infty}}{\sim} \frac{|k|}{2\beta}\, e^{-2t|k|} \tag{5.31}$$

(eq. (80) of [15] with $\pi\rho\delta u^2/\gamma = 2t$). On the other hand we can easily check that

$$\mathrm{Re}\left(\frac{1}{(v+u+2it)^2} + \frac{1}{(v-u-2it)^2}\right) = -2\int_0^\infty dk\, k e^{-2kt}\cos(kv)\cos(ku)\,. \tag{5.32}$$

Taking the inverse transform of this result and comparing with (5.31) shows that the latter formula is equivalent to

$$(4XY)\rho^{T\,\rm hard}_{(1+1)}\left(X^2,Y^2;t\right) \underset{X,Y,t\to\infty}{\sim} -\frac{1}{\pi^2\beta}\mathrm{Re}\left(\frac{1}{(X+Y+2it)^2} + \frac{1}{(X-Y+2it)^2}\right) \tag{5.33}$$

for the leading non-oscillatory behaviour.

The prediction (5.1) can readily be checked on the exact formula (5.29) with $\beta=1,2$ and 4 initial conditions. The simplest case is $\beta=2$ initial conditions, when the product $\tilde{I}^{\rm hard}_\beta \tilde{D}^{\rm hard}_\beta$ in (5.29) vanishes. To analyze $S_2^{\rm hard}$ and $\tilde{S}_2^{\rm hard}$ we make use of the asymptotic expansion (5.17), and furthermore expand the integrands about their endpoint $u=1$. Integration by parts then gives

$$S_2^{\rm hard}\left(X^2,Y^2;0,t\right) \;\sim\; \frac{e^t}{2\pi(XY)^{1/2}}\,\mathrm{Re}\left(\frac{e^{i(X-Y)}}{-i(X-Y)-2t} + \frac{e^{i((X+Y)-\pi\alpha/2-\pi/2)}}{-i(X+Y)-2t}\right), \tag{5.34}$$

$$\tilde{S}_2^{\rm hard}\left(Y^2,X^2;t,0\right) \;\sim\; \frac{e^{-t}}{2\pi(XY)^{1/2}}\,\mathrm{Re}\left(\frac{e^{i(X-Y)}}{i(X-Y)-2t} + \frac{e^{i((X+Y)-\pi\alpha/2-\pi/2)}}{i(X+Y)-2t}\right), \tag{5.35}$$

and these formulas when substituted in (5.29) imply the result (5.33) with $\beta=2$. In the case of $\beta=1$ and $\beta=4$ initial conditions, similar analysis shows

$$\begin{aligned}
&\tilde{I}^{\rm hard}_\beta\left(X^2,Y^2;0,t\right) D^{\rm hard}_\beta\left(X^2,Y^2;0,t\right)\\
&\qquad\sim\; \chi_\beta \tilde{S}^{\rm hard}_\beta\,(X,Y;0,t)\,\tilde{S}^{\rm hard}_\beta\,(Y,X;t,0)\\
&\qquad\sim\; \chi_\beta\frac{1}{4XY}\,\frac{1}{2\pi^2}\mathrm{Re}\left(\frac{1}{(X+Y+2it)^2} + \frac{1}{(X-Y+2it)^2}\right),
\end{aligned} \tag{5.36}$$

where $\chi_\beta = -1$ for $\beta=1$ and $\chi_\beta = 1/2$ for $\beta=4$, and the asymptotics refer to the leading non-oscillatory term. Thus (5.33) again remains valid.

Let us now turn our attention to the same asymptotic limit for the soft edge. There it is known that the asymptotic form (5.33) for $X,Y\to\infty$ is valid in the static theory ($t=0$) [22]. It turns out that the exact results for (5.30) satisfy (5.33) for $X,Y\to\infty$

and $t \to \infty$, as just demonstrated for the hard edge correlations. Thus, for $\beta = 2$ initial conditions, use of the asymptotic expansion (5.24) in (5.7) and (5.8) shows

$$\begin{aligned}
&S_2^{\rm soft}\left(-X^2, -Y^2; 0, t\right) \\
&\qquad \sim \quad \frac{1}{2\pi (XY)^{1/2}} \,\mathrm{Re}\left(\frac{e^{2i(X^3+Y^3)/3 - iv(X+Y) - \pi i/2}}{i(X+Y)+t} + \frac{e^{2i(X^3-Y^3)/3 - iv(X-Y)}}{i(X-Y)+t} \right), \\
&\tilde{S}_2^{\rm soft}\left(-Y^2, -X^2; t, 0\right) \\
&\qquad \sim \quad \frac{1}{2\pi (XY)^{1/2}} \,\mathrm{Re}\left(\frac{e^{2i(X^3+Y^3)/3 - iv(X+Y) - \pi i/2}}{i(X+Y)-t} + \frac{e^{2i(X^3-Y^3)/3 - iv(X-Y)}}{i(X-Y)-t} \right),
\end{aligned}$$

and these formulas when substituted in (5.30) (with $\tilde{I}_2^{\rm soft} D_2^{\rm soft} = 0$) gives the asymptotic form (5.33) (although t therein must be rescaled by replacing $2t$ with t). The same conclusion is reached for $\beta = 1$ and 4 initial conditions, where we find the analogue of the formula (5.36).

As pointed out in [15], a consequence of the asymptotic form (5.33) is a universal formula for the time displaced covariance of two linear statistics

$$A_t = \sum_{j=1}^N a\left(x_j^2(t)\right), \qquad B_t = \sum_{j=1}^N b\left(x_j^2(t)\right).$$

Thus, if in the general formula

$$\mathrm{Cov}(A_0, B_t) = \int_0^\infty dx\, a(x^2) \int_0^\infty dy\, b(y^2)(4xy)\rho_{(1+1)}^T(x^2, y^2; t)$$

we suppose $a(x^2)$ and $b(x^2)$ are made slowly varying by writing

$$a(x^2) \mapsto a((x/\alpha)^2), \qquad b(x^2) \mapsto b((x/\alpha)^2),$$

where $\alpha \gg 1$, and if we scale t according to

$$t \mapsto \alpha\, t,$$

then a simple change of variables and use of (5.33) shows that for $\alpha \to \infty$

$$\begin{aligned}
\mathrm{Cov}(A_0, B_t) \quad &\sim \quad -\frac{1}{\pi^2 \beta} \int_0^\infty dx\, a(x^2) \int_0^\infty dy\, b(y^2) \mathrm{Re}\left(\frac{1}{(X+Y+2it)^2} + \frac{1}{(X-Y+2it)^2} \right) \\
&= \quad \frac{2}{\pi^2 \beta} \int_0^\infty dk\, k\, \tilde{a}(k)\, \tilde{b}(k)\, e^{-2kt}, \qquad (5.37)
\end{aligned}$$

where the equality follows from (5.32), and

$$\tilde{a}(k) := \int_0^\infty a(x^2) \cos(kx)\, dx$$

and similarly the meaning of $\tilde{b}(k)$.

5.5 Empirical density of eigenvalues at the soft edge

The dynamical distribution functions at the soft edge calculated in Section 4 can be realized by parameter dependent random matrices of the type (1.3). Thus we can numerically construct random matrices and empirically compute eigenvalue distributions for comparison against the theoretical prediction. The simplest distribution to compute empirically is the density (one-point function). Here we will undertake such a calculation for the orthogonal to unitary transition at the soft edge.

As deduced from (2.24), (2.26) and (3.31) the theoretical prediction for the scaled density at the soft edge of the orthogonal to unitary transition is

$$\begin{aligned}\rho_{(1)}^{\rm soft}(X;t) &= S_1^{\rm soft}(X,X;t,t) \\ &= \int_0^\infty \Big(\mathrm{Ai}(X+v)\Big)^2\,dv + \frac{1}{2}\int_0^\infty e^{-stx}\,\mathrm{Ai}(X-s)\,ds. \qquad (5.38)\end{aligned}$$

The theory of Section 1 tells us that the parameter dependent random matrices (1.3), for an appropriate $X^{(0)}$ have, in the infinite dimension limit, a scaled density at the soft edge given by (5.38). Since we are considering the orthogonal to unitary transition, $X^{(0)}$ must be chosen as a real symmetric matrix with joint p.d.f. for the elements

$$P(X^{(0)}) = \prod_{j=1}^N \Big(\frac{1}{2\pi}\Big)^{1/2} e^{-X_{jj}^{(0)\,2}/2} \prod_{j<k} \frac{1}{\pi^{1/2}} e^{-X_{jk}^{(0)\,2}}.$$

The joint distribution of the elements of X, $P(X;\tau)$ say, is obtained by integrating over $X_{jj}^{(0)}$ and $X_{jk}^{(0)}$. Thus

$$\begin{aligned}P(X;\tau) &= \prod_{j=1}^N \int_{-\infty}^\infty dX_{jj}^{(0)} \prod_{j<k}\int_{-\infty}^\infty dX_{jk}^{(0)}\, P(X^{(0)})P(X^{(0)};X;\tau) \\ &= \prod_{j=1}^N \frac{e^{-X_{jj}^2/(1+e^{-2\tau})}}{\sqrt{\pi(1+e^{-2\tau})}} \prod_{j<k} \frac{2}{\pi\sqrt{1-e^{-4\tau}}} e^{-2(\mathrm{Re}X_{jk})^2/(1+e^{-2\tau})} e^{-2(\mathrm{Im}X_{jk})^2/(1-e^{-2\tau})}, \qquad (5.39)\end{aligned}$$

which tells us that the diagonal and the real and imaginary parts of the upper triangular elements are all independent, having Gaussian distributions with mean zero and variances $2/(1+e^{-2\tau})$, $4/(1+e^{-2\tau})$ and $4/(1-e^{-2\tau})$ respectively. Hence, for a given value of N and τ such matrices are simple to generate numerically. For each such random matrix the eigenvalues λ_j are computed and scaled according to $\lambda \mapsto (\lambda_j - (2N)^{1/2})2^{1/2}N^{1/6}$. For N large enough and with τ related to t by (4.9) the corresponding empirical density should approach (5.38). The situation for a relative small value of N (recall the occurence of $N^{1/6}$) is illustrated in Figure 1, where good agreement with the infinite dimension limit is observed.

Also of interest is the possibility of providing a realization of the hard edge results. Unfortunately here it seems that the initial conditions used do not precisely correspond to a situation in which X in (1.6) can be generated from a formula analogous to (5.39). The problem is that if $X^{(0)}$ is a real $n \times m$ matrix with independent Gaussian entries, then the square of the eigenvalues of $X^{(0)\dagger}X^{(0)}$ has p.d.f. (1.2) with $a = n-m-1$. However in

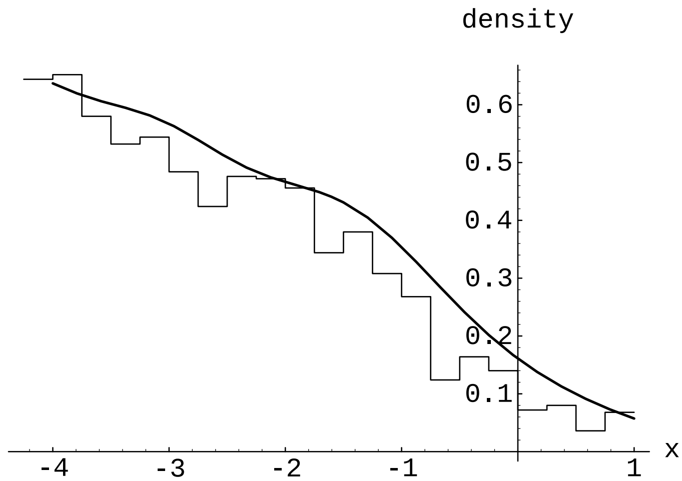

Figure 1: Empirical histogram of the scaled soft edge density for 500 parameter dependent random matrices of the form (5.39) with $\tau = .05$ and $N = 100$, compared against the infinite dimension form (5.38).

deducing (1.4) from (1.6) in the case of transitions to unitary symmetry ($\beta = 2$) we must take $a = n - m$, which is thus incompatible with the sought initial condition.

5.6 Distribution functions in the initial state

With $m = 0$ and $\tau_1 \to 0$ the dynamical distribution (1.8) reduces to the n-point distribution function in the initial state. At $\beta = 1$ the formulas (2.24), (2.30), (2.31) and (2.33) reclaim the well known [12, 10, 6, 23] general formula

$$\rho_{(n)}(x_1, \dots, x_n) = \mathrm{Tdet} \begin{bmatrix} S_1(x_j, x_k) & \frac{1}{2}\int \mathrm{sgn}(x_k - y) S_1(x_j, y)\, dy - \frac{1}{2}\mathrm{sgn}(x_k - x_j) \\ \frac{\partial}{\partial x_j} S_1(x_j, x_k) & S_1(x_k, x_j) \end{bmatrix}, \tag{5.40}$$

where $S_1(x_j, x_k) := S(x_j, x_k; 0, 0)$ as specified by (2.20) with $\{R_j\}$ corresponding to the $\beta = 1$ initial condition. We remark that the integral in (5.40) can be rewritten according to

$$\frac{1}{2}\int \mathrm{sgn}(x_k - y) S_1(x_j, y)\, dy = -\int_{x_j}^{x_k} S_1(x_j, y)\, dy.$$

At $\beta = 4$ these same formulas (with (2.33) replaced by (2.34)) give that

$$\rho_{(n)}(x_1, \dots, x_n) = \mathrm{Tdet} \begin{bmatrix} S_4(x_j, x_k) & \frac{\partial}{\partial x_k} S_1(x_k, x_j) \\ -\int_{x_k}^{x_j} S_4(y, x_k)\, dy & S_4(x_k, x_j) \end{bmatrix}, \tag{5.41}$$

where $S_4(x, y) := S(x, y; 0, 0)$ as specified by (2.20) with $\{R_j\}$ therein corresponding to the $\beta = 4$ initial condition, again in accordance with the known result. In deriving (5.41) from (2.24) we assumed that none of the points $x_1, \dots, x_n$ are coincident; this is necessary because of the delta functions in (2.7) with $\beta = 4$.

We can take the limit $t_X, t_Y \to 0$ in the formulas obtained for $S_1^{\mathrm{edge}}(X, Y; t_X, t_Y)$ (here edge denotes hard or soft) obtained in Section 3 and 4 to obtain the explicit form

of $S_1^{\rm edge}(X,Y)$ in the scaled limit at the hard and soft edges. Thus from (3.31), (3.49), (4.22) and (4.40) we find

$$
\begin{aligned}
S_1^{\rm hard}(X,Y) &= K^{\rm hard}(X,Y) - \frac{J_{a+1}(\sqrt{Y})}{4\sqrt{Y}}\Big(\int_0^{\sqrt{X}} J_{a+1}(u)\,du - 1\Big) && (5.42)\\
S_4^{\rm hard}(X,Y) &= K^{\rm hard}(X,Y) - \frac{1}{4X^{1/2}}J_{a-1}(X^{1/2})\int_0^{Y^{1/2}} J_{a+1}(s)\,ds && (5.43)\\
S_1^{\rm soft}(X,Y) &= K^{\rm soft}(X,Y) + \frac{1}{2}{\rm Ai}(Y)\Big(1 - \int_X^\infty {\rm Ai}(t)\,dt\Big) && (5.44)\\
S_4^{\rm soft}(X,Y) &= K^{\rm soft}(X,Y) - \frac{1}{2}{\rm Ai}(X)\int_Y^\infty {\rm Ai}(s)\,ds && (5.45)
\end{aligned}
$$

where

$$
\begin{aligned}
K^{\rm hard}(X,Y) &:= \frac{X^{1/2}J_{a+1}(X^{1/2})J_a(Y^{1/2}) - Y^{1/2}J_{a+1}(Y^{1/2})J_a(X^{1/2})}{2(X-Y)}\\
K^{\rm soft}(X,Y) &:= \frac{{\rm Ai}(X){\rm Ai}'(Y) - {\rm Ai}(Y){\rm Ai}'(X)}{X-Y}. && (5.46)
\end{aligned}
$$

In obtaining these formulas we have used (5.2) and (5.7) together with the known facts that [16]

$$S_2^{\rm hard}(X,Y) = K^{\rm hard}(X,Y), \quad S_2^{\rm soft}(X,Y) = K^{\rm soft}(X,Y)$$

to rewrite the first of the integrals in the expressions (3.31), (3.49), (4.22) and (4.40). Also, Bessel function identities have been used in obtaining the second term in (5.42) and (5.43), while in the second term of (5.44) use has been made of the definite integral

$$\int_{-\infty}^\infty {\rm Ai}(x)\,dx = 1.$$

We remark that in previous studies [6, 24] expression have been obtained for (5.42) and (5.43) which are of a more complicated structure. Also the expressions obtained in [16] for (5.44) and (5.45) are now seen to be in error.

Some caution must be exercised in applying these results at $\beta = 4$. In particular, the one body weight function used in (2.7) at $\beta = 4$ is that corresponding to $\beta = 2$ in (1.1) and (1.2). To obtain from the above results expressions corresponding to the weight functions as written in (1.1) and (1.2) we must make the replacements

$$S_4^{\rm hard}(X,Y) \mapsto 2S_4^{\rm hard}(4X,4Y)\Big|_{a\to 2a}, \quad S_4^{\rm soft}(X,Y) \mapsto \frac{1}{2^{1/3}}S_4^{\rm soft}(2^{2/3}X, 2^{2/3}Y).$$

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Appendix A

In the case $\tau_x = \tau_y = 0$, the formula (3.9) for S_2 at $\beta = 1$ reads

$$S_2(x,y;0,0) = \sqrt{w(y)}\Big(\frac{\Phi_{N-2}(x;0)}{r_{N/2-1}(0)} - \sqrt{w(x)}\frac{C_{N-1}(x)}{h_{N-1}}\Big)\Big(C_{N-1}(y) - (N-1)R_{N-2}(y)\Big). \quad (A.1)$$

Use of the explicit formulas (3.1) and (3.2) gives

$$C_{N-1}(y) - (N-1)R_{N-2}(y) = -(N-1)!\Big(L^a_{N-1}(y) - \frac{d}{dy}L^a_{N-1}(y)\Big) = (N-1)!\frac{d}{dy}L^a_N(y), \tag{A.2}$$

where the second equality follows from an appropriate Laguerre polynomial identity [13].

For the x-dependent factor in (A.1), we require an explicit formula for $\Phi_{N-2}(x;0)$. Now (2.31) substituted in (2.16) gives

$$\Phi_k(x;0) = \int \mathrm{sgn}(x-y)\sqrt{w(y)}R_k(y).$$

Making use of the explicit formulas (3.2) and (3.3) then gives

$$\frac{\Phi_{N-2}(x;0)}{r_{N/2-1}(0)} = -\frac{1}{4\Gamma(a+N)}\int_0^\infty \mathrm{sgn}(x-y)y^{a/2}e^{-y/2}\frac{d}{dy}L^a_{N-1}(y)\,dy.$$

Integrating by parts, and making use of the Laguerre polynomial identity

$$(y-a)L^a_{N-1}(y) = y\frac{d}{dy}L^a_{N-1}(y) - N\Big(L^a_N(y) - L^a_{N-1}(y)\Big),$$

we see from this that

$$\begin{aligned}\frac{\Phi_{N-2}(x;0)}{r_{N/2-1}(0)} &= \sqrt{w(x)}\frac{C_{N-1}(x)}{h_{N-1}} \\ &\quad + \frac{N}{4\Gamma(a+N)}\int_0^\infty \mathrm{sgn}(x-y)y^{a/2-1}e^{-y/2}\Big(L^a_N(y) - L^a_{N-1}(y)\Big)\,dy.\end{aligned} \tag{A.3}$$

Substituting (A.2) and (A.3) in (A.1) shows

$$\begin{aligned}S_2(x,y;0,0) &= \frac{N!}{4\Gamma(N+a)}\sqrt{w(y)} \\ &\quad \times\Big(\frac{d}{dy}L^a_N(y)\Big)\int_0^\infty \mathrm{sgn}(x-y)y^{a/2-1}e^{-y/2}\Big(L^a_N(y) - L^a_{N-1}(y)\Big)\,dy,\end{aligned} \tag{A.4}$$

which is the formula recently given by Widom [8].

Consider next the expression (3.42) for S_2 at $\beta = 4$ with $\tau_x = \tau_y = 0$. To simplify this expression, we note that the formula in (3.39) for $R_{2j}(x)$ can be written in the simplified form

$$R_{2j}(x) = \frac{1}{a+2j+1}\Big(C_{2j+1}(x) + \frac{1}{2}(2j+1)!x^{-a/2}e^{x/2}\int_0^x t^{a/2}e^{-t/2}\Big(\frac{d}{dt}L_{2j+2}(t)\Big)\,dt\Big). \tag{A.5}$$

This formula can be checked by multiplying both sides by $x^{a/2}e^{-x/2}$ and differentiating. (We also verify that the $x \to \infty$ behaviour of both sides agrees.) Now, according to the definition (2.16) of Φ_k and the second formula in (2.32)

$$\frac{\Phi_{N-2}(x;0)}{r_{N/2-1}(0)} = -\frac{2}{r_{N/2-1}(0)}\frac{d}{dx}\Big(x^{a/2}e^{-x/2}R_{N-2}(x)\Big)$$

Substituting (A.5), and making use of the differentiation formulas

$$y\frac{d}{dy}L_n^a(y) = nL_n^a(y) - (n+a)L_{n-1}^a(y) = (n+1)L_{n+1}^a(y) - (n+a+1-y)L_n^a(y)$$

and the fact that

$$(a+N-1)r_{N/2+1}(0) = h_{N-1} = \Gamma(N)\Gamma(N+a)$$

we readily find

$$\frac{\Phi_{N-2}(x;0)}{r_{N/2-1}(0)} - \sqrt{w(x)}\frac{C_{N-1}(x)}{h_{N-1}} = -\sqrt{w(x)}\frac{N}{\Gamma(N+a)}\frac{(L_N^a(x) - L_{N-1}^a(x))}{x} \tag{A.6}$$

which thus simplifies the x-dependent factor in (3.42). For the y-dependent factor, we see from (A.5) that

$$\sqrt{w(y)}\Big(C_{N-1}(y) - (a+N-1)R_{N-2}(y)\Big) = -\frac{1}{2}(N-1)!\int_0^y t^{a/2}e^{-t/2}\frac{d}{dt}L_N^a(t)\,dt. \tag{A.7}$$

Multiplying (A.6) and (A.7) shows

$$S_2(x,y;0,0) = \frac{N!\sqrt{w(x)}}{2\Gamma(N+a)}\frac{L_N^a(x) - L_{N-1}^a(x)}{x}\int_0^y t^{a/2}e^{-t/2}\frac{d}{dt}L_N^a(t)\,dt, \tag{A.8}$$

in agreement with the formula given recently by Widom [8].

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